Competitive Equilibria in Semi-Algebraic Economies

Felix Kubler[1] Karl Schmedders[2]

[1]Swiss Banking Institute, University of Zurich and Swiss Finance Institute

[2]Institute for Operations Research, Univ. of Zurich and Swiss Finance Institute

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swiss:finance:institute

Motivation

Applied general equilibrium models are ubiquitous in economics

Multiplicity of equilibria is a serious threat to predictions and sensitivity analysis

Sufficient conditions for uniqueness exist but are too restrictive

Uniqueness of equilibrium is often just assumed

Algorithms for solving applied models do not search for more than one equilibrium

Prevalence of multiplicity in “realistically calibrated” models is largely unknown

Our Paper

Develop theoretical foundation for the analysis of multiplicity in GE models

Describe the equilibrium correspondence in simple terms to facilitate fast computation of all equilibria

Univariate Polynomial Representation

Under some mild conditions, for generic parameters ξ every equilibrium $x^* = (x_1^*, x_2^*, \ldots, x_M^*)$ along with a positive number y^* is among the finitely many solutions of

$$\begin{array}{ccccccc} x_1 & - & & & & v_1(\xi; y) & = 0 \\ & & x_2 & - & & v_2(\xi; y) & = 0 \\ & & & \ddots & & \vdots & = 0 \\ & & & & x_M \; - & v_M(\xi; y) & = 0 \\ & & & & & r(\xi; y) & = 0 \end{array}$$

Polynomials v_i in y

Polynomial $r(\xi; y)$ in y; zeros of r easy to find for fixed ξ

Arrow-Debreu Exchange Economy

Standard finite Arrow-Debreu exchange economy

- H agents, $h \in \{1, 2, \ldots, H\}$
- L goods, $l \in \{1, 2, \ldots, L\}$

Each individual h is characterized by

- endowments $e^h \in \mathbb{R}^L_{++}$
- utility function $u^h : \mathbb{R}^L_+ \to \mathbb{R}$

u^h is C^1, strictly increasing, and strictly concave
(and more to come)

Equilibrium

Agents maximize utility

$$\begin{aligned} \max_c \quad & u^h(c) \\ \text{s. t.} \quad & p \cdot (c - e^h) = 0 \end{aligned}$$

Optimal solution $c^h = (c_1^h, c_2^h, \ldots, c_L^h)$

Market-clearing condition

$$\sum_{h \in \mathcal{H}} c_l^h \quad = \quad \sum_{h \in \mathcal{H}} e_l^h, \quad \forall l = 1, \ldots, L$$

Interior Walrasian Equilibrium

Unknown: prices p, allocations c^h, multipliers λ^h

Interior Walrasian equilibrium is a strictly positive solution to

$$\begin{array}{rcll} \partial_c u^h(c^h) - \lambda^h p &=& 0, & h = 1, \ldots, H \\ p \cdot (c^h - e^h) &=& 0, & h = 1, \ldots, H \\ \sum_{h \in \mathcal{H}} (c_l^h - e_l^h) &=& 0, & l = 1, \ldots, L-1 \\ \sum_{l=1}^{L} p_l - 1 &=& 0 & \end{array}$$

Many polynomial equations

Outline

1. Introduction
    - Motivation
    - Equilibrium Correspondence
2. Semi-algebraic Exchange Economies
    - Semi-algebraic Sets and Functions
    - Semi-algebraic Marginal Utility
    - Equilibria and Polynomial Equations
3. Computational Algebraic Geometry
    - Ideals and Varieties
    - The Shape Lemma
4. Equilibria
    - Equilibrium Manifold
    - Multiplicity
    - Applications

Polynomials

Monomial in $x_1, x_2, \ldots, x_n$: $\quad x^\alpha \equiv x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots x_n^{\alpha_n}$

Exponents $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^N$

Polynomial f in the n variables $x_1, x_2, \ldots, x_n$ is a linear combination of finitely many monomials with coefficients in a field $\mathbb{K}$

$$f(x) = \sum_{\alpha \in S} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{K}, \quad S \subset \mathbb{Z}_+^N \text{ finite}$$

Examples of $\mathbb{K}$: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

Semi-Algebraic Sets

Every semi-algebraic subset of $\mathbb{R}^n$ can be written as the finite union of sets

$$\{x \in \mathbb{R}^n : f_1(x) = \cdots = f_l(x) = 0,\ g_1(x) > 0, \ldots, g_m(x) > 0\},$$

where $f_1, \ldots, f_l, g_1, \ldots, g_m$ are polynomials in x (with coefficients in $\mathbb{R}$).

A semi-algebraic set A can be decomposed into a finite union of disjoint semi-algebraic sets $(A_i)_{i=1}^p$ where each A_i is (semi-algebraically) homeomorphic to an open hypercube $(0,1)^{d_i}$ for some $d_i \geq 0$.

The dimension of A is $\dim(A) = \max\{d_1, \ldots, d_p\}$.

Semi-Algebraic Functions

Function $\phi : A \to \mathbb{R}^m$ is semi-algebraic if its graph

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : \ x \in A, \ y = \phi(x)\}$$

is a semi-algebraic subset of $\mathbb{R}^{n+m}$

Lemma: Let $\phi : A \to \mathbb{R}$ be a semi-algebraic function. Then there exists a nonzero polynomial $f(x, y)$ in the variables $x_1, \dots, x_n, y$ such that for every $x \in A$ it holds that $f(x, \phi(x)) = 0$.

Proof of Lemma yields simple construction of f

$$y = \phi(x) \quad \Longrightarrow \quad f(x, y) = f(x, \phi(x)) = 0$$

But $f(x, y) = 0$ may have other solutions $y \neq \phi(x)$

Example

The function $\phi : \mathbb{R}_{++} \to \mathbb{R}$ defined by

$$\phi(x) = \begin{cases} 6 - 2x & 0 < x \leq 2, \\ \frac{1}{x} & 2 < x. \end{cases}$$

is semi-algebraic.

Graph of ϕ

$$\begin{array}{rl} & \{(x, y) \in \mathbb{R} \times \mathbb{R} : \ x \in \mathbb{R}_{++}, \ y = \phi(x)\} \\ = & \{(x, y) : y - (6 - 2x) = 0, x > 0, -x > -2\} \\ \cup & \{(x, y) : y - (6 - 2x) = 0, x - 2 = 0\} \\ \cup & \{(x, y) : yx - 1 = 0, x > 2\} \end{array}$$

$f(x, y) = (y - (6 - 2x))^2 (yx - 1)$ not square-free

$f(x, y) = (y - (6 - 2x)) (yx - 1)$

Marginal Utility

Key assumption:

Marginal utility $\partial_{c_l} u^h : \mathbb{R}^L_+ \longrightarrow \mathbb{R}$ is semi-algebraic

Previous lemma implies:

There exists a nonzero polynomial $m^h_l(c, y)$ such that for all c,

$$m^h_l(c, \partial_{c_l} u^h(c)) = 0$$

Example

Utility

$$u(c) = \begin{cases} 8\,\sqrt{c} & & 0 < c \leq 1, \\ 6c - c^2 & + \, const & 1 < c \leq 2, \\ 4\,\mathit{ln}(c) & + \, const & 2 < c. \end{cases}$$

Marginal utility

$$u'(c) = \begin{cases} \frac{4}{\sqrt{c}} & 0 < c \leq 1, \\ 6 - 2c & 1 < c \leq 2, \\ \frac{4}{c} & 2 < c. \end{cases}$$

Polynomial

$$m(c, y) = (16 - cy^2)(6 - 2c - y)(4 - cy)$$

satisfies $m(c, u'(c)) = 0$ for all $c > 0$

Semi-algebraic Marginal Utility

Most utility functions used in applied research satisfy our assumptions

Semi-algebraic preferences as in Blume and Zame (1992) satisfy the assumptions

Any finite set of observations of demands and prices that can be rationalized by any increasing utility function can be rationalized by a strictly concave semi-algebraic C^1 utility function

Assumptions impose no restrictions on equilibrium multiplicity

Polynomial Equilibrium Equations

Equation (part of first-order conditions)

$$\partial_{c_l} u^h(c^h) - \lambda^h p_l = 0, \quad l = 1, \dots, L$$

transformed into polynomial equation

$$m_l^h(c^h, \lambda^h p) = 0, \quad l = 1, \dots, L$$

Polynomial $F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p)$ with $c^{\mathcal{H}} = (c^1, c^2, \dots, c^H)$

$$F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = \begin{cases} m_l^h(c^h, \lambda^h p), & h = 1, \dots, H, \quad l = 1, \dots, L \\ p \cdot (c^h - e^h), & h = 1, \dots, H \\ \sum_{h \in \mathcal{H}} (c_l^h - e_l^h), & l = 1, \dots, L-1 \\ \sum_l p_l - 1 & \end{cases}$$

Equilibria and Polynomial Equations

All Walrasian equilibria are among the solutions to the system

$$F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = 0.$$

For generic endowments $e^{\mathcal{H}}$ the square matrix

$$\partial_{c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p} \, F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = 0$$

has full rank at all Walrasian equilibria.

Task at hand: describe solution set to $F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = 0$

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Polynomial Ideals

Polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ = set of all polynomials in $x = (x_1, \ldots, x_n)$ with coefficients in some field $\mathbb{K}$

$I \subset \mathbb{K}[x]$ is an ideal,

- if $f, g \in I$, then $f + g \in I$;
- if $f \in I$ and $h \in \mathbb{K}[x]$, then $hf \in I$ for

Ideal generated by $f_1, \ldots, f_k$,

$$I = \{\sum_{i=1}^{k} h_i f_i : h_i \in \mathbb{K}[x]\} = \langle f_1, \ldots, f_k \rangle,$$

Polynomials $f_1, \ldots, f_k$ are basis of I

Complex Varieties

Set of common complex zeros of $f_1, \dots, f_k \in \mathbb{K}[x]$

$$V(f_1, f_2, \dots, f_k) = \{x \in \mathbb{C}^n : f_1(x) = f_2(x) = \ldots = f_k(x) = 0\}$$

$V(f_1, f_2, \dots, f_k)$ complex variety defined by $f_1, f_2, \dots, f_k$

Study of polynomial equations on algebraically closed fields

Field $\mathbb{R}$ is not algebraically closed, but $\mathbb{C}$ is

For an ideal $I = \langle f_1, \dots, f_k \rangle = \langle g_1, \dots, g_l \rangle$

$$V(I) = V(f_1, f_2, \dots, f_k) = V(g_1, g_2, \dots, g_l).$$

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Simple Version of the Shape Lemma

$V(f_1, f_2, \ldots, f_n) = \{x \in \mathbb{C}^n : f_1(x) = f_2(x) = \ldots = f_n(x) = 0\}$
is zero-dimensional and has d complex roots
No multiple roots
All roots have distinct value for last coordinate x_n

Then:

$V(f_1, f_2, \ldots, f_n) = V(G)$ where

$$G = \{x_1 - v_1(x_n), x_2 - v_2(x_n), \ldots, x_{n-1} - v_{n-1}(x_n), r(x_n)\}$$

Polynomial r has degree d, polynomials v_i have degrees less than d

On the Assumptions

$x_1^2 - x_2 = 0, \; x_2 - 4 = 0$ has solutions $(2,4), (-2,4)$

No polynomial $x_1 - v_1(x_2)$ can yield 2 and -2 for $x_2 = 4$

After reordering of variables the shape lemma holds
$x_2 - 4 = 0, x_1^2 - 4 = 0$

$x_1^2 + x_2 - 1 = 0, \; x_2^2 - 1 = 0$, sol's $(\sqrt{2}, -1), (-\sqrt{2}, -1), (0, 1)$

Solution $(0,1)$ has multiplicity 2
No linear term in x_1 of form $x_1 - v_1(x_2)$ can yield multiplicity

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Buchberger's Algorithm

Example of Gröbner basis
$G = \{x_1 - v_1(x_n), x_2 - v_2(x_n), \ldots, x_{n-1} - v_{n-1}(x_n), r(x_n)\}$

Buchberger's algorithm allows calculation of Gröbner bases

If all coefficients of $f_1, \ldots, f_n$ are rational then the polynomials r, $v_1, v_2, \ldots, v_{n-1}$ have rational coefficients and can be computed exactly!

Software SINGULAR: implementation of Buchberger's algorithm

Parameterized Shape Lemma

Let $\Xi \subset \mathbb{R}^m$ be an open set of parameters and let $f_1, \ldots, f_n \in \mathbb{K}[\xi_1, \ldots, \xi_m; x_1, \ldots, x_n]$ with $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$ and $(x_1, \ldots, x_n) \in \mathbb{C}^n$. Suppose that for each $\bar{\xi} = (\bar{\xi}_1, \ldots, \bar{\xi}_m) \in \Xi$ the Jacobian matrix $D_x f(\bar{\xi}; x)$ has full rank n whenever $f(\bar{\xi}; x) = 0$ and all d solutions have a distinct last coordinate x_n.

Then there exist $r, v_1, \ldots, v_{n-1} \in \mathbb{K}[\xi; x_n]$ and $w_1, \ldots, w_{n-1} \in \mathbb{K}[\xi]$ such that for generic $\bar{\xi}$,

$$\begin{aligned} &\{x \in \mathbb{C}^n : f_1(\bar{\xi}; x) = \ldots = f_n(\bar{\xi}; x) = 0\} = \\ &\Big\{x \in \mathbb{C}^n : w_1(\bar{\xi})x_1 = v_1(\bar{\xi}; x_n), \ldots, w_{n-1}(\bar{\xi})x_{n-1} = v_{n-1}(\bar{\xi}; x_n); \\ &\qquad\qquad r(\bar{\xi}; x_n) = 0\Big\}. \end{aligned}$$

The degree of r in x_n is d, the degrees of $v_1, \ldots, v_{n-1}$ in x_n are at most $d - 1$.

Theorem

For generic endowments, $e^{\mathcal{H}} \in \mathbb{R}^{HL}_{++}$, every competitive equilibrium x^* of the economy along with a positive number y^* is among the finitely many common zeros of the polynomials in a set $\mathcal{G}$ of the shape

$$\mathcal{G} = \left\{x_1 - v_1(e^{\mathcal{H}}; y), \ \ldots \ , \ x_M - v_M(e^{\mathcal{H}}; y), \ r(e^{\mathcal{H}}; y)\right\}.$$

The non-zero polynomial $r \in \mathbb{R}[e^{\mathcal{H}}; y]$ is not constant in y. Each v_i, $i = 1, \ldots, M$, is a polynomial in y of degree less than the degree of r. The coefficients of this polynomial are rational functions of $e^{\mathcal{H}}$.

Proof

Our polynomial system $F(e^{\mathcal{H}}; c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = 0$

Zero-dimensional, d complex roots, no multiple roots?
perhaps not; add a variable t and a new equation

$$1 - t \; det[\partial F] = 0$$

No Walrasian equilibria lost (for generic endowments)

All roots have distinct value for last variable?
perhaps not; add a variable y and a new equation

$$y - \sum_i \mu_i x_i = 0 \quad \text{with} \sum_i \mu_i = 1, \mu_i > 0 \quad (x = (c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p))$$

Proof: Applying the Shape Lemma

For generic weights μ and endowments the system

$$\begin{array}{rcl} F(e^{\mathcal{H}}; x) & = & 0 \\ 1 - t \; det[\partial F(e^{\mathcal{H}}; x)] & = & 0 \\ y - \sum_i \mu_i x_i & = & 0 \end{array}$$

satisfies the assumptions of the Parameterized Shape Lemma

Gröbner basis

$$G = \{x_1 - v_1(e^{\mathcal{H}}; y), x_2 - v_2(e^{\mathcal{H}}; y), \ldots, x_n - v_n(e^{\mathcal{H}}; y), \\ t - v_{n+1}(e^{\mathcal{H}}; y), r(e^{\mathcal{H}}; y)\}$$

All Walrasian equilibria among the solutions

Theorem for General Parameter Sets

Suppose that for generic parameters $\xi \in \Xi$, every Walrasian equilibrium satisfies

$$\det \left[D_{c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p} F(\xi; c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) \right] \neq 0.$$

Then for generic $\xi \in \Xi$, every Walrasian equilibrium x^* of the economy along with an accompanying positive number y^* is among the finitely many common zeros of the polynomials in a set $\mathcal{G}$ of the shape

$$\mathcal{G} = \{x_1 - v_1(\xi; y), \ \ldots \ , \ x_M - v_M(\xi; y), \ r(\xi; y)\} . \qquad (1)$$

The non-zero polynomial $r \in \mathbb{R}[\xi; y]$ is not constant in y. Moreover, each v_i, $i = 1, \ldots, M$, is a polynomial in y of degree less than the degree of r. The coefficients of this polynomial are rational terms in ξ.

Real Solutions

Univariate polynomial $r(x_n) = \sum_{i=0}^{d} a_i \; x_n^i$ of degree d

Fundamental Theorem of Algebra

Polynomial $r(x_n)$ has d complex roots.

Bounds on the number of (positive) real roots exist

Descartes's Rule of Signs

The number of positive reals roots of a polynomial is at most the number of sign changes in its coefficient sequence

Sturm's Theorem gives exact number of real zeros

in a given interval

Summary: Solving Polynomial Systems

Objective: find all solutions to $f(x) = 0$ with $x, f(x) \in \mathbb{R}^n$

View the system in complex space, $f(x) = 0$ with $x, f(x) \in \mathbb{C}^n$

$$V(f_1, f_2, \ldots, f_n) = \{x \in \mathbb{C}^n : f_1(x) = f_2(x) = \ldots = f_n(x) = 0\}$$

Apply Buchberger's algorithm to find Gröbner basis G

If Shape Lemma holds, then $V(f) = V(G)$ for a G of the shape

$$G = \{x_1 - v_1(x_n), x_2 - v_2(x_n), \ldots, x_{n-1} - v_{n-1}(x_n), r(x_n)\}$$

Apply Sturm's Theorem to r to find number of real solutions

Find approximation of all (complex) solutions by solving $r(x_n) = 0$

CES Utility

Marginal utility of good l for agent h with $\sigma_h \in \mathbb{N}$

$$u'_{hl}(c) = (\alpha_l^h)^{-\sigma_h}(c_l)^{-\sigma_h}$$

In polynomial form

$$m_l^h(c, y) = (\alpha_l^h)^{\sigma_h}(c_l)^{\sigma_h} y - 1 = 0$$

First-order condition

$$\alpha_{hl}^{-\sigma^h}(c_l^h)^{-\sigma_h} - \lambda^h p_l = 0$$

Rewritten

$$\alpha_{hl}^{\sigma^h}(c_l^h)^{\sigma_h}\lambda^h p_l - 1 = 0$$

Economy is parameterized by endowments e_l^h and weights α_{hl}

Identical CES Utility

$H = 2$ agents and $L = 2$ goods

Univariate polynomial r has the shape

$$r(e^{\mathcal{H}}, \alpha^{\mathcal{H}}) = C_4 \ y^{\sigma} - C_3 \ y^{\sigma-1} + C_2 \ y - C_1$$

with constants $C_4, C_3, C_2, C_1 > 0$

Number of sign changes is 3: $+ - + -$

At most three equilibria, bound is tight

Incomplete Markets

Two time periods, $t = 0, 1$, one state in second period

$s = 0, 1$, one good in $s = 0$, two goods in $s = 1$

Heterogenous quadratic utility functions

Five endowments are fixed, $e^1_{11} = e$ is parameter

Single real asset paying $A(p_{11}, p_{12}) = p_{12} - p_{11}$

Drop in rank at $p_{12} = p_{11}$

Computing equilibria notoriously difficult

Gröbner Basis

$$\begin{aligned} r(y,e) &= 3058149y^3 + (46572e - 6197797)y^2 \\ &\quad +(335e^2 - 100175e + 3233873)y \\ &\quad +(e^3 - 359e^2 + 18625e - 24176) \\ \theta^1 &= \frac{C_1}{D}y^2 + \frac{C_2}{D}y + \frac{C_3}{D} \end{aligned}$$

with

$$\begin{aligned} D &= 9e^5 - 7175e^4 + 1623387e^3 - 77212929e^2 \\ &\quad +243727714e - 191476296 \end{aligned}$$

Basis is not well-defined for $D = 0$ at $e \in \{1.332, 2, 59.6\}$

For these parameter values $p_{12} = p_{11}$ $(= 1)$

Bad Prices

Gröbner Basis is only generically correct

For $e = 2$ (no-trade) equilibrium does exist, but
Gröbner Basis does not properly specialize

For $e \in \{1.332, 59.6\}$ there is no equilibrium

Non-existence is no obstacle to Buchberger's algorithm

Gröbner Basis exists, just has no solutions that are equilibria

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Summary

Develop theoretical foundation for the analysis of multiplicity in GE models

Describe the equilibrium correspondence in simple terms to facilitate fast computation of all equilibria

Identify assumption on preferences to obtain polynomial equilibrium conditions:

Marginal utilities are continuous semi-algebraic functions

Apply methods from computational algebraic geometry, Gröbner bases and Buchberger's algorithm

Summary cont'd

Show that all equilibria are solutions to a simple system of polynomial equations

Mention how this univariate polynomial is derived exactly, without making approximations, numerical errors

Outline how the methods apply to more complex GE models

Outlook

Systems of polynomial equations are ubiquitous in economics

Methods from algebraic geometry are widely applicable

We have already computed multiple

- equilibria in infinite-horizon models with complete markets
- stationary equilibria (steady states) in OLG model
- Nash equilibria in strategic market games
- perfect Bayesian equilibria in a game with cheap talk