

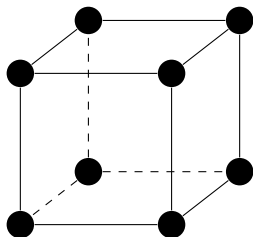
Combinatorial abstractions for the diameter of polyhedra

Nicolai Hähnle

EPFL DISOPT

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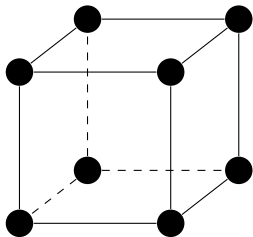
Diameter of polyhedra



- ▶ Look at pointed polyhedra of fixed dimension d and number of facets n
- ▶ The diameter of the edge-vertex graph is believed to be bounded by a linear function in n and d
- ▶ Best known general upper bounds are superpolynomial
- ▶ Hirsch conjecture (diameter $\leq n - d$) is known to be true for $d \leq 3$ and many special classes of polyhedra
 - ▶ there are counter-examples in the unbounded case
 - ▶ counter-examples still have diameter $< n$

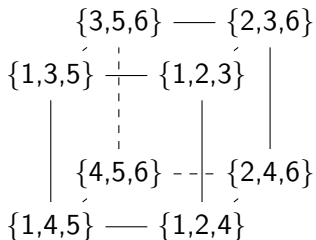
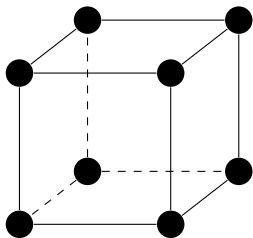
Abstracting (simple, pointed) polyhedra

Facets become symbols, each vertex is identified with the set of facets that contain it.



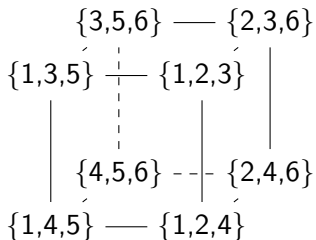
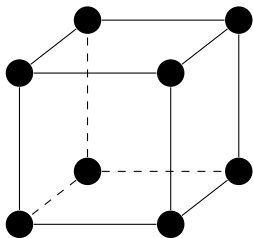
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$$\Rightarrow \{ \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \\ \{2, 3, 6\}, \{2, 4, 6\}, \{3, 5, 6\}, \{4, 5, 6\} \}$$

Ultraconnected families of d -sets

Definition (Kalai)

S a set of n symbols, d -element subsets of S called *vertices*.

To a family V of vertices we associate a graph $G(V) = (V, E)$ by

$$\{u, v\} \in E \iff |u \cap v| = d - 1$$

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V is **ultraconnected** if

- ▶ for all $u, v \in V$ there exists a path $u = v_0, v_1, \dots, v_r = v$ in G such that $(u \cap v) \subset v_j$ for all j .

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Theorem (Kalai & Kleitman)

$$\delta(V) \leq n^{1+\log d}$$

Abstract polytopes

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V is an *abstract polytope* if in addition

- ▶ any set of $d - 1$ symbols is contained in no vertices of V or in exactly two

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Theorem (Klee & Walkup)

If $n - d \leq 5$, then $\delta(V) \leq n - d$ for abstract polytopes.

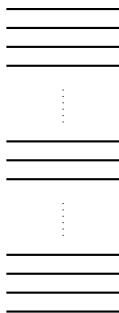
Ultraconnected layer systems

Given a family V of vertices, a partition of V into layers

$$\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_\ell)$$

is an *ultraconnected layer system* if

- ▶ for all $1 \leq i < j < k \leq \ell$ and for all $u \in \mathcal{L}_i$, $v \in \mathcal{L}_k$ there exists a $w \in \mathcal{L}_j$ such that $(u \cap v) \subset w$



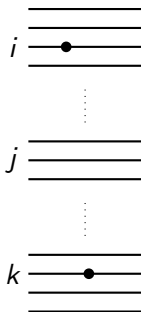
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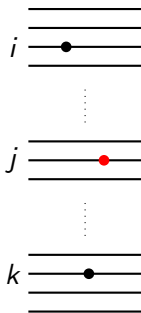
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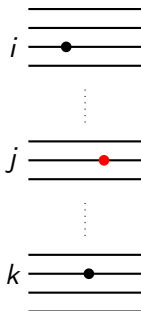
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Equivalently,

- ▶ all subsets of symbols are **active** in a contiguous, possibly empty sequence of layers



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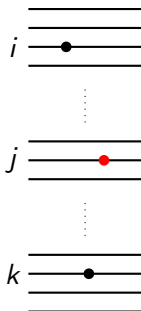
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d is the *dimension* of \mathcal{L}

$\delta(\mathcal{L}) := \ell$ its *diameter*

$\Delta(d, n)$ is the largest diameter of a d -dimensional ultraconnected layer system on n symbols



Ultraconnected family \rightarrow ultraconnected layer system

Let V be an ultraconnected family,

$v_0, v'_0 \in V$ such that $d(v_0, v'_0) = \delta(V) =: \ell$.

Define layers $\mathcal{L}_j = \{v \in V \mid d(v_0, v) = j\}$, $j = 0 \dots \ell$

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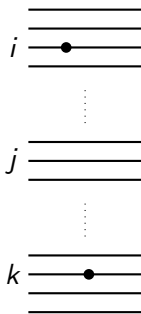
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- ▶ ultraconnectedness of V guarantees a path from u to v that “stays on” $u \cap v$
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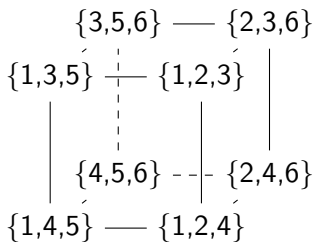
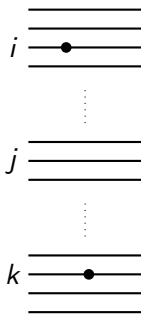
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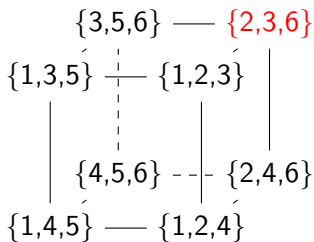
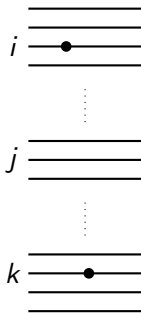
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\mapsto

$$\mathcal{L}_0 = \{\{2, 3, 6\}\}$$

$$\mathcal{L}_1 = \{\{3, 5, 6\}, \{1, 2, 3\}, \{2, 4, 6\}\}$$

$$\mathcal{L}_2 = \{\{1, 3, 5\}, \{4, 5, 6\}, \{1, 2, 4\}\}$$

$$\mathcal{L}_3 = \{\{1, 4, 5\}\}$$

Upper bounds

Induction on a symbol

- ▶ Facets of a d -dim. polyhedron are $(d - 1)$ -dim. polyhedra
- ▶ Given an ULS \mathcal{L} of dimension d and n symbols
- ▶ Given a symbol s that is active in \mathcal{L} , construct $\mathcal{L}[s]$
 1. Remove vertices that do not contain s
 2. Remove s from all vertices
 3. Remove empty layers
- ▶ $\mathcal{L}[s]$ is an ULS of dimension $d - 1$ and $n - 1$ symbols

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A pseudo-polynomial upper bound

Theorem (Kalai & Kleitman)

$$\Delta(d, n) \leq n^{1+\log d}$$

Proof.

Let $(\mathcal{L}_1, \dots, \mathcal{L}_\ell)$ be a d -dim. ULS with n symbols.

Let L be the largest integer such that at most $\frac{n}{2}$ symbols are active on layers $\mathcal{L}_1, \dots, \mathcal{L}_L$.

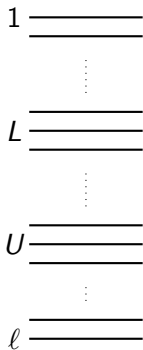
Let U be the smallest integer such that at most $\frac{n}{2}$ symbols are active on layers $\mathcal{L}_U, \dots, \mathcal{L}_\ell$.

There is a symbol s that is active in $\mathcal{L}_{L+1}, \dots, \mathcal{L}_{U-1}$.

$$\Delta(d, n) \leq 2\Delta(d, \frac{n}{2}) + \Delta(d-1, n)$$

$$\leq 2 \sum_{j=2}^d \Delta(j, \frac{n}{2}) + \Delta(1, n)$$

$$\leq \dots \leq n^{1+\log d} \quad \square$$



A linear bound in n (for constant d)

Theorem

$$\Delta(d, n) \leq 2^{d-1}n$$

Proof.

Another induction on d , but let's skip this. □

Lower bounds

A relaxation

Vertices are now d -element *multisets* of symbols.

$$\{\{1, 1, 1\}\}$$

$$\{\{1, 1, 2\}\}$$

$$\{\{1, 2, 2\}\}$$

$$\{\{2, 2, 2\}\}$$

$$\{\{2, 2, 3\}\}$$

\vdots

$$\{\{n - 1, n, n\}\}$$

$$\{\{n, n, n\}\}$$

$\implies \Delta(d, n) \geq d(n - 1) + 1$ (for this relaxation!)

Is $\Delta(d, n)$ for multisets significantly larger than for sets?

How to construct large d -dimensional ULS

Symbols: $S = S_1 \cup S_2 \cup \dots \cup S_k$, $|S_j| = m$, pairwise disjoint

Bridge(S_1)

Mesh($S_1, 2; S_2, 1$)

Mesh($S_1, 1; S_2, 2$)

Bridge(S_2)

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Mesh($S_{k-1}, 1; S_k, 2$)

Bridge(S_k)

How to construct large d -dimensional ULS

Symbols: $S = S_1 \cup S_2 \cup \dots \cup S_k$, $|S_j| = m$, pairwise disjoint

► We want:

Bridge(S_1)

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Bridge(S_2)

Mesh($S_2, 2; S_3, 1$)

⋮

Mesh($S_{k-1}, 1; S_k, 2$)

Bridge(S_k)

► *Bridge*(S_j) satisfies:

- vertices v are d -element subsets of S_j
- every $(d - 1)$ -subset of S_j is active on all layers

► *Mesh*($S_j, i; S_{j+1}, d - i$) satisfies:

- vertices v are d -element sets with $|v \cap S_j| = i$ and $|v \cap S_{j+1}| = d - i$
- all proper subsets of all such sets are active on all layers

► Every component has almost m layers

► get an ULS with $n = km$ symbols and almost $m(d(k - 1) + 1) \approx dn$ layers

A toy example

Parameters are: $d = 2$, $k = 2$, $m = 4$

$Bridge(S_1)$	$\{1, 2\}$	$\{3, 4\}$		
	$\{1, 3\}$	$\{2, 4\}$		
	$\{1, 4\}$	$\{2, 3\}$		
<hr/>				
$Mesh(S_1, 1; S_2, 1)$	$\{1, 5\}$	$\{2, 6\}$	$\{3, 7\}$	$\{4, 8\}$
	$\{1, 6\}$	$\{2, 7\}$	$\{3, 8\}$	$\{4, 5\}$
	$\{1, 7\}$	$\{2, 8\}$	$\{3, 5\}$	$\{4, 6\}$
	$\{1, 8\}$	$\{2, 5\}$	$\{3, 6\}$	$\{4, 7\}$
<hr/>				
$Bridge(S_2)$	$\{5, 6\}$	$\{7, 8\}$		
	$\{5, 7\}$	$\{6, 8\}$		
	$\{5, 8\}$	$\{6, 7\}$		

How to construct a bridge

Bridge(S_j) satisfies:

- ▶ vertices v are d -element subsets of S_j
 - ▶ every $(d - 1)$ -subset of S_j is active on all layers
- \implies each layer is an $(m, d, d - 1)$ -covering of S_j

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Definition

Let $n > k > r$ be natural numbers and let X be a set, $|X| = n$. A collection C of k -subsets of X is called an (n, k, r) -covering of X if every r -subset of X is contained in at least one k -subset in C .

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The *disjoint covering number*, $DC(n, k, r)$, is the maximum size of a family of disjoint (n, k, r) -coverings.

We are interested in $DC(m, d, d - 1)$.

Simple counting argument $\implies DC(m, d, d - 1) \leq m - d + 1$.

What we really need is a good lower bound!

Disjoint Coverings

Theorem

$$DC(m, d, d - 1) \geq m - d^2 + 1$$

Proof.

- ▶ Use integers modulo m as set of symbols.
- ▶ Create preliminary collections of d -element sets

$$C_j = \{A \subset \mathbb{Z}_m \mid |A| = d, \sum_{a \in A} a = j\} \text{ for } j = 0 \dots m - 1$$

- ▶ Every $(d - 1)$ -set is covered in exactly $m - d + 1$ of the C_j .
- ▶ Use the Marriage Theorem to fill “holes” in $m - d^2 + 1$ of the C_j , using one d -set from the other $d^2 - 1$ collections per “hole”. □

How to construct a mesh

- ▶ $\text{Mesh}(S_j, i; S_{j+1}, d - i)$ satisfies:
 - ▶ vertices v are d -element sets with $|v \cap S_j| = i$ and $|v \cap S_{j+1}| = d - i$
 - ▶ all proper subsets of all such sets are active on all layers
- ▶ can be constructed by combining
 - ▶ a family of disjoint $(m, i, i - 1)$ -coverings of S_j and
 - ▶ a family of disjoint $(m, d - i, d - i - 1)$ -coverings of S_{j+1}

Generalized construction

Theorem

Given a d -dimensional multiset-ULS with n symbols and diameter δ , there is a d -dimensional ULS with nm symbols and diameter $\delta(m - d^2 + 1)$.

Corollary

For every $d \geq 2$ there are infinitely many n such that $\Delta(d, n) \geq dn - 3d^2\sqrt{n}$.

Corollary

Allowing d to grow as a function of n , we get a lower bound of $\Omega(n^{3/2-\varepsilon})$ for every $\varepsilon > 0$.

Open Problems

- ▶ Find better upper/lower bounds for $\Delta(d, n)$
 - ▶ What about $\Delta(3, n)$?
 - ▶ We know $3n - c\sqrt{n} \leq \Delta(3, n) \leq 4n$.
 - ▶ Conjecture: If you can prove $\Delta(3, n) \leq 3n$, you can prove $\Delta(d, n) \leq dn$.
 - ▶ Can we say anything when $|\mathcal{L}_j|$ is bounded?
 - ▶ Can we say anything about ULS that contain a small number of vertices?
 - ▶ Can we say anything about ULS that contain a large number of vertices?
 - ▶ We know that the diameter is $\leq d(n - d) + 1$ for complete ULS
- ▶ Investigate the relationship of ultraconnected families and ultraconnected layer systems.
- ▶ Relationship to combinatorial abstractions in other areas?
- ▶ Useful algorithmic problem on ULS? (B. Gärtner)