Proof. For $t = n$ the assertion follows from Lemma 9.6 and Theorem 9.1. Then we proceed by downward induction. By induction hypothesis, the inequalities

$$d^T x - r \sum_{j \in N_0 \cup \{t+1\}} x_j - r \sum_{j \in N_1} (1 - x_j) \leq d_0,$$
$$d^T x - r \sum_{j \in N_0} x_j - r \sum_{j \in N_1 \cup \{t+1\}} (1 - x_j) \leq d_0$$

are DCs for P_{t+1}, which is exactly the setup of Proposition 9.3. It follows that

$$d^T x - r \sum_{j \in N_0} x_j - r \sum_{j \in N_1} (1 - x_j) \leq d_0$$

is a DC for P_t. ∎

Proof. (of Theorem 9.5) Lemmas 9.6 and 9.7 imply that every valid inequality for P_I is valid for P_0. Conversely, every point of S is contained in P_0. Hence $P_0 = P_I$. Finally, let $d^T x \leq d_0$ be valid for S. But $\mathrm{DC}(d, d_0, 0, r, \emptyset, \emptyset)$ is exactly $d^T x \leq d_0$, and it is a DC. ∎

9.5 All valid inequalities for IPs

All polyhedra we consider in this section will be subsets of the positive orthant, i.e., the set $\{x : x \geq \mathbf{0}\}$, even if the theory can be developed for general polyhedra. Our assumption is convenient mainly because of our definition of dominated inequalities.

Definition 9.8 Let $P = \{x \in \mathbb{R}^n : x \geq \mathbf{0},\ Ax \leq b\}$. An inequality $dx \leq d_0$ valid for the integer hull P_I is

▷ a **Chvátal rank 0 inequality** if it is equivalent to or dominated by a non-negative linear combination of the inequalities $Ax \leq b$ defining P;

▷ a **Chvátal rank t inequality** if it is not a Chvátal rank t' inequality for any $t' < t$ but it is equivalent to or dominated by a non-negative linear combination of inequalities of the form
$$\sum_{j=1}^{n} \lfloor u^T g_j \rfloor x_j \leq \lfloor u^T h \rfloor,$$
where $Gx \leq h$ is a system of inequalities of Chvátal rank at most $t-1$, g_j is the jth column of G and $u \geq \mathbf{0}$ (cf. Proposition 9.2);

▷ a **Chvátal–Gomory inequality** (**CGI**) if it is a Chvátal rank t inequality for some t.

Let $P^t = \{x \in \mathbb{R}^n : x \geq \mathbf{0},\ x$ satisfies all inequalities of Chvátal rank at most t for $P\}$. Clearly $P^0 = P$, $P_I \subseteq P^t$ and $P^{t+1} \subseteq P^t$ for all t. Our goal is to prove:

Theorem 9.9 (Schrijver (1980)) *For every rational polyhedron $P = \{x \in \mathbb{R}^n : x \geq \mathbf{0}$, $Ax \leq b\}$ there exists a non-negative integer t such that $P_I = P^t$.*

Corollary 9.10 *Every valid inequality for the integer hull of a rational polyhedron is a CGI.*

Recall from Section 5.6 that a rational system $Ax \le b$ is **totally dual integral** if for any integral vector c, if the LP $\min\{b^T y : y^T A = c,\ y \ge \mathbf{0}\}$ has an optimal solution, it has an integral optimal solution.

Lemma 9.11 *Suppose $P = \{x \in \mathbb{R}^n : x \ge \mathbf{0},\ Ax \le b\}$ is a polyhedron with A integral and $x \ge \mathbf{0}$, $Ax \le b$ totally dual integral. Then $P^1 = P^* := \{x \in \mathbb{R}^n : x \ge \mathbf{0},\ Ax \le \lfloor b \rfloor\}$, where $\lfloor b \rfloor$ is the vector whose ith component is $\lfloor b_i \rfloor$.*

Proof. Each inequality among $Ax \le \lfloor b \rfloor$ has Chvátal rank at most 1; thus $P^1 \subseteq P^*$. Now consider $\bar{x} \in P^*$ and $u \ge \mathbf{0}$. We want to show that

$$\sum_{j=1}^{n} \left\lfloor u^T a_j \right\rfloor \bar{x}_j \le \left\lfloor u^T b \right\rfloor .$$

Observe that

$$\sum_{j=1}^{n} \left\lfloor u^T a_j \right\rfloor x_j \le u^T A x \le u^T b$$

for any $x \in P$. Hence

$$u^T b \ge \max\Big\{ \sum_{j=1}^{n} \left\lfloor u^T a_j \right\rfloor x_j : x \ge \mathbf{0},\ Ax \le b \Big\}$$
$$= \min\left\{ b^T y : y \ge \mathbf{0},\ y^T a_j \ge \left\lfloor u^T a_j \right\rfloor \text{ for all } j \right\} . \quad (9.7)$$

Because of total dual integrality, the minimization problem has an integral optimal solution y^*. Therefore

$$\begin{aligned}
\sum_{j=1}^{n} \left\lfloor u^T a_j \right\rfloor \bar{x}_j &\le y^{*T} A \bar{x} && \text{because } y^{*T} a_j \ge \left\lfloor u^T a_j \right\rfloor \text{ for all } j \\
&\le y^{*T} \lfloor b \rfloor && \text{because } \bar{x} \in P^* \\
&\le \left\lfloor y^{*T} b \right\rfloor && \text{because } y^* \ge \mathbf{0} \\
&\le \left\lfloor u^T b \right\rfloor && \text{by (9.7).}
\end{aligned}$$

∎

Lemma 9.12 *For any rational polyhedron P, P^1 is a polyhedron.*

Proof. By a theorem of Giles and Pulleyblank (1979), for every rational polyhedron P there is a totally dual integral system $Ax \le b$ with A integral such that $P = \{x : Ax \le b\}$. Then the claim follows from Lemma 9.11. ∎

Lemma 9.13 *If F is a face of a rational polyhedron P, then $F^t = F \cap P^t$.*

Proof. We prove that $F^1 = F \cap P^1$; the rest follows by induction. By the same theorem of Giles and Pulleyblank (1979), $P = \{x \in \mathbb{R}^n : Ax \le b\}$ with A integral and $Ax \le b$ totally dual integral. A face $F = \{x \in \mathbb{R}^n : Ax \le b,\ d^T x = d_0\}$, where $d^T x \le d_0$ is some valid inequality for P, $d \in \mathbb{Z}^n$, $d_0 \in \mathbb{Z}$. It can be shown that the system $Ax \le b$, $d^T x = d_0$ is totally dual integral; thus by Lemma 9.11,

$$F^1 = \left\{x \in \mathbb{R}^n : Ax \le \lfloor b \rfloor\,,\ d^T x \le \lfloor d_0 \rfloor\,,\ d^T x \ge \lceil d_0 \rceil\right\} = \left\{x \in \mathbb{R}^n : Ax \le \lfloor b \rfloor\,,\ d^T x = d_0\right\},$$

whereas $P^1 = \{x \in \mathbb{R}^n : Ax \le \lfloor b \rfloor\}$, and so $F \cap P^1 = \left\{x \in \mathbb{R}^n : Ax \le \lfloor b \rfloor\,,\ d^T x = d_0\right\}$. ∎

Proof. (of Schrijver's Theorem 9.9) By induction on the dimension d of P. If $d = -1$ ($P = \emptyset$) or $d = 0$ ($P = \{x\}$), the claim is trivial.

For higher dimensions, we have already observed that $P_I \subseteq P^t$ for all t; thus it remains to show that $P^t \subseteq P_I$ for some t. We distinguish three cases:

1. *The affine hull* $\operatorname{aff}(P)$ *contains no integral vectors.* Then $\operatorname{aff}(P)$ is contained in a hyperplane $H = \{x \in \mathbb{R}^n : a^T x = b\}$ with $H \cap \mathbb{Z}^n = \emptyset$ and a integral; the inequalities $a^T x \le b$ and $a^T x \ge b$ are valid for P, hence (using Lemma 9.1)

$$P^1 \subseteq \left\{x \in \mathbb{R}^n : a^T x \ge \lceil b \rceil\,,\ a^T x \le \lfloor b \rfloor\right\} = \emptyset = P_I.$$

2. $\operatorname{aff}(P)$ *contains integral vectors but* P *is not full-dimensional.*
By Hermite normal form theory and because the theorem is invariant under integral translations, we may assume that $\operatorname{aff}(P) = \left\{x \in \mathbb{R}^n : \begin{bmatrix} B & \mathbf{0} \end{bmatrix} x = \mathbf{0}\right\}$ for some invertible B, hence $\operatorname{aff}(P) = \{0\}^{n-d} \times \mathbb{R}^d$ and $P = \{0\}^{n-d} \times P'$ for some full-dimensional polyhedron $P' \subseteq \mathbb{R}^d$. Since $P_I = \{0\}^{n-d} \times P'_I$ and $P^t = \{0\}^{n-d} \times (P')^t$, we may assume that:

3. *P is full-dimensional.* It is not difficult to show that there exists an integral $m \times n$ matrix A and $b, b' \in \mathbb{Q}^m$ such that $P = \{x \in \mathbb{R}^n : Ax \le b\}$ and $P' = \{x \in \mathbb{R}^n : Ax \le b'\}$.
Claim. *Every inequality $a_i^T x \le b'_i$ from the system $Ax \le b'$ is valid for P^{t_i} for some non-negative integer t_i.*
If we can establish the claim, let $t = \max\{t_i : i = 1, \ldots, m\}$; then $P^t \subseteq P_I$.
Proof of Claim. By contradiction. Assume that $a_i^T x \le b'_i$ is not valid for any P^s. The inequality $a_i^T x \le b_i$ has Chvátal rank 0 for P, thus $a_i^T x \le \lfloor b_i \rfloor$ has Chvátal rank at most 1. Hence there exists $b''_i \in \mathbb{Z}$ such that $b'_i < b''_i \le \lfloor b_i \rfloor$ and for all sufficiently large s, the inequality $a_i^T x \le b''_i$ is valid for P^s but $a_i^T x \le b''_i - 1$ is *not* valid for P^s.
In particular, the set $\{x \in \mathbb{R}^n : a_i^T x = b''_i\}$ contains no integral vectors; nor does $F = P^s \cap \{x \in \mathbb{R}^n : a_i^T = b''_i\}$, whose dimension is at most $n - 1$.
If $F = \emptyset$, then $P^s \subseteq \{x \in \mathbb{R}^n : a_i^T x < b''_i\}$, and so $P^{s+1} \subseteq \{x \in \mathbb{R}^n : a_i^T x \le b''_i - 1\}$, a contradiction. Hence F is a nonempty face of P^s. By induction, $F^r = \emptyset$ for some non-negative integer r. Therefore

$$P^{s+r} \cap \{x \in \mathbb{R}^n : a_i^T x = b''_i\} = P^{s+r} \cap F = F^r = \emptyset,$$

and so $P^{s+r} \subseteq \{x \in \mathbb{R}^n : a_i^T x < b''_i\}$ and $P^{s+r+1} \subseteq \{x \in \mathbb{R}^n : a_i^T x \le b''_i - 1\}$, again a contradiction.

∎

Another consequence of Theorem 9.9 is the following earlier result:

Theorem 9.14 (Chvátal (1973)) *For each polytope P there exists a non-negative integer t such that $P_I = P^t$.*

Here, the assumption of rationality is replaced with boundedness.

The **Chvátal rank of a polyhedron** P is the maximum Chvátal rank of an inequality valid for P_I. Theorem 9.9 implies that every polyhedron has finite Chvátal rank.

There is an interesting connection between the complexity of a (combinatorial) optimization problem and the Chvátal ranks of arising polyhedra. For instance, in Section 9.2, we considered the polyhedra arising from (IP3), the integer program to compute a maximum matching in a graph. Edmonds (1965) showed that all these so-called *matching polyhedra* have Chvátal rank at most 1. On the other hand, Boyd and Pulleyblank (1984) proved that unless NP = co-NP, if the optimization problem over the polyhedra arising from an NP-hard problem can be solved in polynomial time, then their Chvátal rank is not bounded by any constant.