9 Valid inequalities

Most of our exposition in this chapter is based on the book [9]; Section 9.5 follows [11, §23.1].

9.1 Valid inequalities for polyhedra

An inequality $d^T x \le d_0$ is **valid** for a set $S \subseteq \mathbb{R}^n$ if $d^T x \le d_0$ for all $x \in S$. The inequalities $d^T x \le d_0$ and $d'^T x \le d'_0$ are **equivalent** if $(d, d_0) = \lambda(d', d'_0)$ for some $\lambda > 0$. If they are not equivalent but there is $\lambda > 0$ such that $d' \ge \lambda d$ and $d'_0 \le \lambda d_0$, then $\{x : x \ge \mathbf{0},\ d'^T x \le d'_0\} \subset \{x : x \ge \mathbf{0},\ d^T x \le d_0\}$ and we say that $d'^T x \le d'_0$ **dominates** $d^T x \le d_0$. If a valid inequality is not dominated by any other valid inequality, it is called a **maximal valid inequality**.

Note. Any maximal valid inequality for S defines a nonempty face of $\operatorname{conv}(S)$. Any facet-defining inequality of $\operatorname{conv}(S)$ is a maximal valid inequality for S.

In the context of integer programming, we will be interested in valid inequalities for $S = P \cap \mathbb{Z}^n$ for a polyhedron P; since the valid inequalities for S and for $P_I = \operatorname{conv} S$ are the same, we use the two terms interchangeably. First we investigate valid inequalities for P.

Theorem 9.1 *Let $P = \{x \in \mathbb{R}^n : Ax \le b,\ x \ge \mathbf{0}\} \neq \emptyset$, $A \in \mathbb{R}^{m \times n}$, and let $d^T x \le d_0$ be a valid inequality for P. Then there exists $u \in \mathbb{R}^m$, $u \ge \mathbf{0}$, with at most $\min\{m, n\}$ non-zero components such that $d^T x \le d_0$ is equivalent to or dominated by $(u^T A)x \le u^T b$.*

Proof. The linear program $D = \max\{d^T x : x \ge \mathbf{0},\ Ax \le b\}$ is feasible because $P \neq \emptyset$ and bounded because $D \le d_0$. Therefore the dual $\min\{b^T u : u \ge \mathbf{0}, u^T A \ge d\}$ is also feasible and bounded. A basic optimal solution u proves the claim. ∎

Note. Theorem 9.1 characterizes all valid inequalities if $P \neq \emptyset$, that is, if the *primal* LP $\max\{d^T x : x \ge \mathbf{0},\ Ax \le b\}$ is feasible. A similar result applies if the *dual* LP $\min\{b^T u : u \ge \mathbf{0}, u^T A \ge d\}$ is feasible. However, if both the primal and the dual are infeasible, then $P = \emptyset$ and so *any* inequality is valid for P. They cannot, however, all be generated as linear combinations of the defining inequalities $Ax \le b$. Therefore it is sometimes assumed that $A = \left[\begin{smallmatrix} A' \\ I \end{smallmatrix}\right]$, which provides an implicit bounding box for the primal problem and makes the dual feasible.

Next we look at some methods to generate valid inequalities for P_I.

9.2 Integer rounding

The main idea is that if $a \le b$ and a is an integer, then $a \le \lfloor b \rfloor$.

Recall from Definition 8.11 that a matching in a graph is a set M of edges such that no vertex is contained in more than one edge from M. A maximum matching is a matching of largest cardinality. By Proposition 8.12, a maximum matching in a graph $G = (V, E)$ can be computed by solving the integer program

$$\alpha'(G) = \max\Big\{\mathbf{1}^T x : \forall i \in V,\ \sum_{\substack{e \in E \\ i \in e}} x_e \le 1;\ x \ge \mathbf{0};\ x \in \mathbb{Z}^m\Big\}. \tag{IP3}$$

Just like we did in Section 5.2, we can argue that within any set S of vertices, there may be at most $|S|/2$ matching edges. Hence for any $S \subseteq V$ the inequality

$$\sum_{\substack{e \in E \\ e \subseteq S}} x_e \leq \left\lfloor \frac{|S|}{2} \right\rfloor \tag{9.1}$$

is valid for

$$P_I = \operatorname{conv}\Big\{ x \in \mathbb{Z}^m : \forall i \in V,\ \sum_{\substack{e \in E \\ i \in e}} x_e \leq 1;\ x \geq \mathbf{0} \Big\}.$$

In the RHS of (9.1) we may write $\lfloor |S|/2 \rfloor$ instead of $|S|/2$ because the LHS is an integer.

Is there a way to get (9.1) just from the inequalities of (IP3)? Let $S \subseteq V$ and take a linear combination of the inequalities with coefficients $u_i = 1/2$ for $i \in S$ and $u_i = 0$ for $i \in V \setminus S$. Thus we obtain

$$\sum_{\substack{e \in E \\ e \subseteq S}} x_e + \frac{1}{2} \sum_{\substack{e \in E \\ |e \cap S| = 1}} x_e \leq \frac{|S|}{2}. \tag{9.2}$$

Since all $x_e \geq 0$, we have

$$-\frac{1}{2} \sum_{\substack{e \in E \\ |e \cap S| = 1}} x_e \leq 0;$$

adding to (9.2) yields

$$\sum_{\substack{e \in E \\ e \subseteq S}} x_e \leq \frac{|S|}{2}. \tag{9.3}$$

Finally, as before, since the LHS of (9.3) is an integer, we get (9.1). In this case, taking inequalities 9.1 for all subsets $S \subseteq V$ determines P_I; this is not true for general IPs.

Proposition 9.2 *Let* $P = \{x \in \mathbb{R}^n : x \geq \mathbf{0},\ Ax \leq b\}$, $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$. *Then for any* $u \geq \mathbf{0}$*, the inequality*

$$\sum_{j=1}^{n} \left\lfloor u^T a_j \right\rfloor x_j \leq \left\lfloor u^T b \right\rfloor \tag{9.4}$$

is valid for $P \cap \mathbb{Z}^n$.

Proof. First,

$$\sum_{j=1}^{n} u^T a_j x_j \leq u^T b$$

is valid because it is a non-negative linear combination of the valid inequalities $Ax \leq b$. Moreover,

$$-\left(u^T a_j - \left\lfloor u^T a_j \right\rfloor\right) x_j \leq 0$$

for every j. Hence

$$\sum_{j=1}^{n} \left\lfloor u^T a_j \right\rfloor x_j \leq u^T b$$

is valid. Finally,

$$\sum_{j=1}^{n} \lfloor u^T a_j \rfloor x_j \le \lfloor u^T b \rfloor$$

is valid, because the LHS is an integer. ∎

An inequality of the form (9.4) is called a **Chvátal–Gomory inequality** (**CGI**).

9.3 Gomory cutting planes

Let $P = \{x \in \mathbb{R}^n : x \ge \mathbf{0},\ a^T x = b\}$ and let $S = \{x \in \mathbb{Z}^n : x \ge \mathbf{0},\ a^T x - b \in \mathbb{Z}\}$. Clearly,

$$S = \Big\{x \in \mathbb{Z}^n : x \ge \mathbf{0},\ \sum_{i=1}^{n} (a_i - \lfloor a_i \rfloor) x_i - (b - \lfloor b \rfloor) \in \mathbb{Z}\Big\}.$$

Since $\sum_{i=1}^{n} (a_i - \lfloor a_i \rfloor) x_i \ge 0$ and $b - \lfloor b \rfloor < 1$, we get that

$$\sum_{i=1}^{n} (a_i - \lfloor a_i \rfloor) x_i \ge b - \lfloor b \rfloor \tag{9.5}$$

is valid for S. Because $P \cap \mathbb{Z}^n \subseteq S$, (9.5) is also valid for $P \cap \mathbb{Z}^n$. An inequality of the form (9.5) is called a **Gomory cutting plane** (**GCP**).

Example 9.1 1. If

$$\frac{7}{5}x_1 - \frac{3}{4}x_2 + \frac{1}{4}x_3 - \frac{2}{5}x_4 = \frac{19}{5}$$

and the variables are required to be non-negative integers, then

$$\frac{2}{5}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{3}{5}x_4 \ge \frac{4}{5}.$$

2. If

$$13x_1 - 5x_2 + 19x_3 = 35$$

and the variables are non-negative integers, then

$$\frac{13}{6}x_1 - \frac{5}{6}x_2 + \frac{19}{6}x_3 = \frac{35}{6}$$

and thus

$$\frac{1}{6}x_1 + \frac{1}{6}x_2 + \frac{1}{6}x_3 \ge \frac{5}{6},$$
$$x_1 + x_2 + x_3 \ge 5.$$

9.4 Disjunctive constraints

Example 9.2 Let $P = \{x \in \mathbb{R}^n : x \geq \mathbf{0},\ -2x_1 + 2x_2 \leq 3,\ 2x_1 + x_2 \leq 3\}$. The constraints defining P are equivalent to

$$x_1 + 2x_2 - 3x_1 \leq 3, \tag{9.6a}$$

$$x_1 + 2x_2 + 3(x_1 - 1) \leq 3. \tag{9.6b}$$

If $x_1 \in \mathbb{Z}$, then $x_1 \leq 0$ or $x_1 \geq 1$. In the former case, (9.6a) implies that $x_1 + 2x_2 \leq 3$; in the latter case (9.6b) implies that $x_1 + 2x_2 \leq 3$. Hence $x_1 + 2x_2 \leq 3$ is valid for $P \cap \mathbb{Z}^2$.

Proposition 9.3 *Let $S \subseteq \mathbb{Z}^n$. If r, s are positive numbers such that $d^T x + r(x_i - \ell) \leq d_0$ is valid for S and $d^T x - s(x_i - \ell + 1) \leq d_0$ is valid for S, then $d^T x \leq d_0$ is valid for S.*

Proof. Exercise. ∎

Definition 9.4 Let $S = \{x \in \mathbb{Z}^n : Ax \leq b\}$. **Disjunctive constraints** (**DC**s) for S are defined recursively as follows:

1. All the inequalities of $Ax \leq b$ are DCs.
2. A non-negative linear combination of DCs is a DC.
3. An inequality obtained from DCs using Proposition 9.3 is a DC.
4. An inequality equivalent to or dominated by a DC is a DC.

Next we consider 0, 1-programs. Let $S = \{x \in \{0,1\}^n : Ax \leq b\}$, $P_I = \operatorname{conv} S$, $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. Our goal is the following theorem.

Theorem 9.5 *Let $S = \{x \in \{0,1\}^n : Ax \leq b\}$. Then every valid inequality for S is a DC.*

In fact, P_I is determined by inequalities in a special form. For an inequality $d^T x \leq d_0$ valid for S, $t \in \{0, 1, \ldots, n\}$, $r \geq 0$ and N_0, N_1 such that $N_0 \cap N_1 = \emptyset$, $N_0 \cup N_1 = \{1, 2, \ldots, t\}$, let $\mathrm{DC}(d, d_0, t, r, N_0, N_1)$ be the inequality

$$\sum_{j=1}^{n} d_j x_j - r \sum_{j \in N_0} x_j - r \sum_{j \in N_1} (1 - x_j) \leq d_0. \tag{DC(d, d_0, t, r, N_0, N_1)}$$

Lemma 9.6 *If $d^T x \leq d_0$ is valid for S, then there exists $r \geq 0$ such that all the inequalities $\mathrm{DC}(d, d_0, n, r, N_0, N_1)$ for all partitions (N_0, N_1) of $\{1, \ldots, n\}$ are valid for P.*

Proof. If $P = \emptyset$, then any inequality is valid for P. Otherwise P is nonempty and bounded, and so it suffices to show that each such inequality is valid for the vertices of P. Determining the right value of r is left as an exercise. ∎

Let $P_n = P$ and let $P_t = \operatorname{conv}\Big(\big(P_{t+1} \cap \{x : x_{t+1} = 0\}\big) \cup \big(P_{t+1} \cap \{x : x_{t+1} = 1\}\big)\Big)$ for $t = 0, 1, \ldots, n - 1$.

Lemma 9.7 *If $d^T x \leq d_0$ is valid for S, then there is some $r \geq 0$ such that every inequality $\mathrm{DC}(d, d_0, t, r, N_0, N_1)$ is a DC for P_t.*

10 References

For the theory of integer programming, excellent sources are [11, 3, 13]. Our lecture uses many of the materials given in these books.

Combinatorial optimization problems yields important models of integer programming. Some of the excellent books on combinatorial optimization are [6, 7, 12]. There are a few classical books, e.g. [8, 10] that are still very useful.

There is an open source software for mixed integer programming [2], while CPLEX is a very efficient commercial MIP solver. Another useful software related to integer programming is 4TI2 [1] which can be used to, for example, Hilbert bases of polyhedral cones.

References

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