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Chapter 2

PRICING AMERICAN PUT OPTIONS BY FAST SOLUTIONS OF THE LINEAR COMPLEMENTARITY PROBLEM

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Abstract The value function of an American put option defined in a discrete domain may be given as a solution of a Linear Complementarity Problem (LCP). However, the state of the art methods that solve LCP converge slowly. Recently, Dempster, Hutton & Richards have proposed a Linear Program (LP) formulation of the American put and a special simplex algorithm that exploits the option structure. They give numerical examples with run times which grow almost linearly with the number of spatial grid points. Based on these ideas we show in a constructive fashion that a new algorithm may be devised which processes the original LCP in linear number of spatial grid points.

Keywords: American option pricing, Linear Complementarity, Early exercise, PDE discretization

1. Introduction

The fast valuation of American options has been a long standing problem [Rogers and Talay, 1997]. Computing tools of practitioners must be able to give an answer within seconds or minutes. Popular methods like dynamic programming, Monte Carlo and projected successive over-relaxation (PSOR) techniques are rather slow. For more than one underlying security the only method is Monte Carlo and the problem becomes itself a challenge in scientific computing [Avramidis *et al*, 2000].

For European options linear algorithms exist. But for the American exercise style an optimization problem must be solved on top of the Black-Scholes partial differential equation (PDE) cast in the form of a linear complementary problem (LCP). Compared to European options this is a complication of algorithmic nature. A fast solution of LCP is therefore critical for the whole PDE solver and the complexity of the overall pricing. The general pivoting algorithms are too slow. Fortunately the discretization leads to special class of Z-matrices for which the algorithm of Chandrasekaran converges in polynomial time [Cottle and Pang and Stone, 1992]. Other discretization schemes lead to different types of matrices for which there exist often a corresponding algorithm [Huang and Pang, 1998].

In fact the most popular algorithm so far has been the PSOR iterative method of Cryer [Cryer, 1971]. The method is the usual SOR method for solving linear systems which is modified to update only non-negative SOR solutions. But the number of iterations required to converge is usually large.

Only recently, a new algorithm was proposed by Dempster, Hutton & Richards [Dempster, Hutton and Richards, 1998] which evaluates the American option in an apparent linear time. The authors show the equivalence of the LCP to the corresponding LP. To solve the latter problem they make two plausible assumptions on the form of the complementary basis.

The LCP is solved by formally proving that a complementary feasible basis alluded above exists. The corresponding algorithm that finds it is provided. In the next section the notation used is described and define the problem of evaluation of the American option as a sequence of LCPs. In section 3 the proof and the algorithm that solves the LCPs in linear time are shown. The conclusions follow in section 4.

2. Definition of the pricing problem

The notation used in [Dempster, Hutton and Richards, 1998] is adopted. Let us assume a Black-Scholes economy with one risky asset price S modeled by a geometric Brownian motion with constant volatility σ and a savings account with constant risk-free rate $r \geq 0$.

An European option gives the holder the right to buy or sell one unit of the asset for a price K, the *strike price*, at the *maturity* date T. In contrast, an American option can be exercised at any time τ to maturity, i.e. $\tau \in [0,T]$. The payoff of an American put option is a function $\psi : \mathbb{R}^+ \to \mathbb{R}$ defined by:

$$\psi(S_\tau) = (K - S_\tau)^+.$$

The value function $v : \mathbb{R}^+ \times [0,T] \to \mathbb{R}$ is the "fair" value $v(x,t)$ at asset price $x > 0$ and at time $t \in [0,T]$. It can be formulated as the solution of an optimal stopping problem, namely choose the stopping time which maximizes the conditional expectation of the discounted payoff. The stopping time may be shown to be the first time the value falls to the payoff at exercise [Myneni, 1992]. In particular, the (x,t) domain may be partitioned as follows:

$$\begin{aligned} C &= \{(x,t) \in \mathbb{R}^+ \times [0,T) : v(x,t) > \psi(x)\} \quad \text{and} \\ S &= \{(x,t) \in \mathbb{R}^+ \times [0,T) : v(x,t) = \psi(x)\}. \end{aligned}$$

On the *continuation region* C, v has to satisfy the Black-Scholes PDE:

$$(L_{BS} + \partial_t)v = 0 \;\; \text{with} \;\; v > \psi, \tag{2.1}$$

whereas on the *stopping region* S one avoids arbitrage by requiring:

$$(L_{BS} + \partial_t)v \leq 0 \;\; \text{with} \;\; v = \psi,$$

with L_{BS}, the Black-Scholes operator defined by:

$$L_{BS} = \frac{\sigma^2}{2} x^2 \partial_x^2 + rx\partial_x - r. \tag{2.2}$$

Conditions (2.1-2) lead to the following order complementarity problem (OCP) for the fair value of the American put option [Borwein and Dempster, 1989]:

$$\begin{aligned} &\text{OCP} \\ &\text{Find } \; v \in F \;\; \text{such that:} \\ &-(L_{BS} + \partial_t)v \wedge (v - \psi) = 0, \\ &\text{with} \\ &F \; = \{v : v - \psi \geq 0, -(L_{BS} + \partial_t)v \geq 0\}, \end{aligned} \tag{2.3}$$

where $\wedge$ denotes the point-wise minimum of two functions with respect to a *vector lattice* Hilbert space (see [Borwein and Dempster, 1989] for further discussion).

Note that the Black-Scholes PDE is a linear elliptic PDE with non-constant coefficients. In fact, a log-transformed stock price $\xi = \log x$ is useful to define a path-independent Black-Scholes operator:

$$L_{BS} = \frac{\sigma^2}{2} \partial_\xi^2 + (r - \frac{\sigma^2}{2})\partial_\xi - r, \tag{2.4}$$

with a terminal condition (corresponding to the payoff function) given by:

$$\psi(\xi, T) = (K - e^{\xi})^{+}. \tag{2.5}$$

Hereafter is will be assumed that the above form of the Black-Scholes operator (2.4) and terminal condition (2.5).

Formulation on a discrete and finite domain

Since the analytical solution to the above OCP (2.3) is not known one resorts to numerical methods. For a numerical approximation the function space has to be finite and the value function discrete.

We define the problem on a rectangular domain $[L,U] \times [0,T]$ and assume that the infinite domain solution is recovered in the limit $L \to -\infty, U \to +\infty$. In order to maintain path-independence, the differential operators are approximated by homogeneous *finite differences* on a lattice with $(I+1) \times (M+1)$ number of points. We label these points by indices as follows:

$$\begin{aligned} &\xi_i = L + i\Delta\xi, \quad i = 0, \dots, I, \quad \Delta\xi = (U-L)/I \quad \text{and} \\ &t_m = T - m\Delta t, \quad m = 0, \dots, M, \quad \Delta t = T/M. \end{aligned}$$

The discrete value function on this domain is denoted by:

$$v_i^m = v(\xi_i, t_m), \quad m = 0, \dots, M, \quad i = 0, \dots, I,$$

with *boundary values*:

$$v_0^m = \psi(L), \;\; v_I^m = \psi(U), \quad m = 0, \dots, M,$$

and *terminal value*:

$$v_i^0 = \psi(\xi_i) \equiv \psi_i, \quad i = 0, \dots, I.$$

The time derivative in the Black-Scholes equation (2.1) is approximated by the finite difference:

$$\partial_t v \approx \frac{v_i^{m-1} - v_i^m}{\Delta t}.$$

Spatial derivatives in the Black-Scholes operator (2.4) are approximated by finite difference derivatives:

$$\begin{aligned} &\partial_\xi v \approx \theta \frac{v_{i+1}^m - v_{i-1}^m}{2\Delta\xi} + (1-\theta)\frac{v_{i+1}^{m-1} - v_{i-1}^{m-1}}{2\Delta\xi} \quad \text{and} \\ &\partial_\xi^2 v \approx \theta \frac{v_{i+1}^m - 2v_i^m + v_{i-1}^m}{(\Delta\xi)^2} + (1-\theta)\frac{v_{i+1}^{m-1} - 2v_i^{m-1} + v_{i-1}^{m-1}}{(\Delta\xi)^2}, \end{aligned}$$

whereas the value function near the constant term is split as follows:

$$v \to \theta v_i^m + (1-\theta) v_i^{m-1}.$$

Here $i = 1,\ldots,I-1$ and $m = 1,\ldots,M$ and θ is a parameter that controls the stability of the proposed difference scheme. The scheme is shown to be unconditionally stable for $0.5 \le \theta \le 1$. For the financial oriented reader we refer to [Tavella and Randall, 2000].

Let $n = I-1$ and $v \in \mathbb{R}^n$ be the discrete value function in vector notations:

$$v^m = \begin{pmatrix} v_1^m \\ v_2^m \\ \vdots \\ v_n^m \end{pmatrix}.$$

Then terminal and boundary value vectors $\psi, \phi \in \mathbb{R}^n$ are given by:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix} \quad \text{and} \qquad \phi = \rho(1-\tilde{q}) \begin{pmatrix} \psi_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where ρ is called *the mesh ratio* and given by:

$$\rho = \frac{\sigma^2 \Delta t}{(\Delta\xi)^2}.$$

We define also the matrix $\tilde{Q} \in \mathbb{R}^{n\times n}$:

$$\tilde{Q} = \begin{pmatrix} 0 & \tilde{q} & & \\ 1-\tilde{q} & 0 & \ddots & \\ & \ddots & \ddots & \tilde{q} \\ & & 1-\tilde{q} & 0 \end{pmatrix},$$

with the matrix element $\tilde{q}$ is given by:

$$\tilde{q} = \frac{1}{2} + \frac{1}{2}(r - \frac{\sigma^2}{2})\frac{\Delta\xi}{\sigma^2}.$$

Then using the Black-Scholes equation (2.1) and above definitions one gets the **Black-Scholes difference equations** as the following sequence of matrix equations:

$$B\, v^{m-1} + A\, v^m - \phi = 0, \quad m = 1,\ldots,M, \tag{2.6}$$

where

$$A = \mathbb{1} + \theta\, [r\Delta t\ \mathbb{1} + \rho(\mathbb{1} - \tilde{Q})] \quad \text{and}$$
$$B = -\mathbb{1} + (1-\theta)\, [r\Delta t\ \mathbb{1} + \rho(\mathbb{1} - \tilde{Q})],$$

with $\mathbb{1} \in \mathbb{R}^{n \times n}$ denoting the identity matrix.

We note first that the matrix A is positive definite, since the symmetric part of $\mathbb{1} - \tilde{Q}$:

$$\begin{pmatrix} 1 & -\frac{1}{2} & & \\ -\frac{1}{2} & 1 & \ddots & \\ & \ddots & \ddots & -\frac{1}{2} \\ & & -\frac{1}{2} & 1 \end{pmatrix},$$

is symmetric positive definite. In the case $0 \leq \tilde{q} \leq 1$ the matrix A has all its off-diagonal elements non-positive and belongs to the class of the so called Z-matrices, a crucial property that we shall use in the next section.

In terms of the problem parameters and the spatial lattice spacing the matrix A will be of Z-type if:

$$|r - \frac{\sigma^2}{2}| \leq \frac{\sigma^2}{\Delta\xi},$$

which we shall assume.

Note that in this case $\tilde{q}$ may be interpreted as the "would be" binomial probability of a small upwards change of the stock price.

With the discrete Black-Scholes equation (2.6) at hand one can formulate the computation of the American option as the following sequence of (linear) complementarity problems:

$$\begin{array}{l} \text{For} \quad m = 1, \ldots, M: \\ v^m \geq \psi \quad \text{and} \\ B\, v^{m-1} + A\, v^m - \phi \geq 0 \quad \text{and} \\ (B\, v^{m-1} + A\, v^m - \phi) \wedge (v^m - \psi) = 0. \end{array} \tag{2.7}$$

where $\wedge$ denotes the componentwise minimum of two vectors $\in \mathbb{R}^n$.

But the above complementarity problems can be written as a usual LCP sequence. Let $u^m \equiv v^m - \psi, m = 1, \ldots, M$ be the excess value vector and $s^m \equiv B\, v^{m-1} + A\, v^m - \phi$ the *slack* vector. Then for $m = 1, \ldots, M$ the time slices of an American option can be computed by solving the following sequence of LCPs:

$$\begin{array}{l} \text{LCP} \\ \text{For} \quad m = 1, \ldots, M: \\ Au^m - s^m = b^m \quad \text{and} \\ u^m \geq 0, \;\; s^m \geq 0, \;\; (s^m)^T u^m = 0, \end{array} \tag{2.8}$$

where

$$b^m = \phi - B(u^{m-1} + \psi) - A\,\psi, \quad u^o = 0,$$

or

$$\begin{array}{l} b^m = b^o - Bu^{m-1} \quad \text{with} \\ b^o = \phi - B\,\psi - A\,\psi \quad \text{and} \\ u^o = 0. \end{array} \tag{2.9}$$

3. Solution of LCP

In the sequel we will investigate the solution to the LCP (2.8). Here we refer to the huge literature on LCP which was summarized by Cottle, Pang and Stone [Cottle and Pang and Stone, 1992].

Since A is a positive definite matrix, it is a so-called P-matrix (a matrix with all its principal minors positive, see [Cottle and Pang and Stone, 1992]) and therefore LCP has a *unique* solution for all right hand sides b^m [Murty, 1972]. Given the existence of the solution to (2.8), we are left with the problem of computing it efficiently.

In fact A is by construction a Z-matrix. In this case the LCP can be solved by pivoting techniques in $O(n^3)$, i.e. polynomial by the method developed by Chandrasekaran [Chandrasekaran, 1970]. In the following we make use of the fact that if A is a Z-matrix the following statements are equivalent (see [Fiedler and Ptak, 1962]):

$$\begin{array}{ll} \text{(a)} & A \text{ is a P-matrix}. \\ \text{(b)} & A^{-1} \geq 0. \\ \text{(c)} & \text{There is } x \geq 0 \text{ s.t. } Ax > 0 \text{ has a solution}. \end{array} \tag{2.10}$$

We will use these equivalent properties in the design of a linear algorithm to solve the LCP as stated below.

Recently Dempster and Hutton used the fact that for Z-matrices a solution to the LCP can be obtained using the least element property as shown in [Dempster and Hutton, 1999]. In particular they noticed that the following sequence of LPs can be solved:

$$\begin{array}{l} \text{LP} \\ \text{For } \; m = 1, \ldots, M : \\ \min c^T u^m, \quad \text{s.t.} \\ Au^m - s^m = b^m \quad \text{and} \\ u^m \geq 0, \; s^m \geq 0, \end{array} \tag{2.11}$$

where $c \in \mathbb{R}^n$ is an arbitrary positive vector. In their algorithm a special form of the optimal feasible basis to (2.11) is assumed without giving a formal proof. It contains slack and real basic variables in the following order:

$$\overline{u}^m_{n_b} = (s^m_1, \ldots, s^m_{n_b}, u^m_{n_b+1}, \ldots, u^m_n)^T. \tag{2.12}$$

Before stating the results presented here the above assumption (2.12) on the complementary feasible basis deserves some comments. The partition of the domain of the American option price in continuation and stopping regions may not only serve to formulate the pricing problem by the OCP (2.3) but to state also that the optimal feasible basis to (2.3) has a continuous analogy of (2.12). Namely for any given time, there exists a point, say x_t, such that the solution $v(x,t)$ coincides with $\psi(x,t)$ for $x \leq x_t$ and is greater than $\psi(x,t)$ for

$x > x_t$ [Myneni, 1992]. Nevertheless a mathematical property that holds in the continuous level does not always carry over to a discrete level, unless the discretization is sufficiently “good”.

This assumption is proved for the *implicit discretization scheme*, i.e. for $\theta = 1$. The main result of this work is the following:

Theorem. *The solution to the LCPs (2.8) are unique and the complementary feasible bases are of the structure of (2.12).*

We give below a constructive proof which serves at the same time as an algorithm that finds the solution to LCP.

According to (2.12) and for a fixed time step m we have the following complementary partition of the LCP (2.8):

$$\begin{pmatrix} A_{11} & -\beta e_{n_b-1} & \\ -\gamma e_{n_b-1}^T & \alpha & -\beta e_1^T \\ & -\gamma e_1 & A_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix} - \begin{pmatrix} s_1 \\ s_2 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{2.13}$$

where the time slice index m has been omitted from the vectors to simplify notations. Note also that by definition of A we have $\alpha = 1 + \theta(r\Delta t + \rho)$, $\beta = \theta\rho\tilde{q}$ and $\gamma = \theta\rho(1 - \tilde{q})$.

Then the following property (which is proved later) holds:

Lemma. *For each time step there is a partition of the above form such that:*

$$b_1, \quad b_2 < 0, \quad A_{33}^{-1} b_3 \geq 0, \tag{2.14}$$

for small enough $\Delta\xi$.

Now we proceed with the proof of the theorem.

Proof of Theorem. The complementary basic solution corresponding to the above partition (2.13) with the property (2.14) reads:

$$\begin{aligned} s_1^{\text{old}} &= -b_1 > 0 \quad \text{and} \\ s_2^{\text{old}} &= -(b_2 + \beta e_1^T u_3^{\text{old}}) \quad \text{and} \\ u_3^{\text{old}} &= A_{33}^{-1} b_3 \geq 0. \end{aligned}$$

Note that if $s_2^{\text{old}} \geq 0$ then the LCP (2.8) is solved for this m. Else $s_2^{\text{old}} < 0$ and by decreasing n_b with one unit one has:

$$b_1^{\text{new}}, \quad b_2^{\text{new}} < 0, \quad b_3^{\text{new}} = \begin{pmatrix} b_2 \\ b_3 \end{pmatrix}$$

The new complementary basic solution in terms of the old one reads:

$$\begin{aligned} s_1^{\text{new}} &= -b_1^{\text{new}} > 0 \\ s_2^{\text{new}} &= -(b_2^{\text{new}} + \beta z) \\ u_3^{\text{new}} &= \begin{pmatrix} z \\ u_3^{\text{old}} + z\gamma A_{33}^{-1} e_1 \end{pmatrix}. \end{aligned} \tag{2.15}$$

with

$$z = \frac{-s_2^{\text{old}}}{\alpha - \beta\gamma e_1^T A_{33}^{-1} e_1}. \tag{2.16}$$

We show now that $u_3^{\text{new}} \geq 0$. First, we observe that $A_{33}^{-1} e_1 \geq 0$ by virtue of property (2.10b). Furthermore the denominator of (2.16) is positive as a principal minor of A being positive definite:

$$\alpha - \beta\gamma e_1^T A_{33}^{-1} e_1 = \det\begin{pmatrix} \alpha & -\beta e_1^T \\ -\gamma e_1 & A_{33} \end{pmatrix} \det A_{33}^{-1}.$$

Also as we noted above $s_2^{\text{old}} < 0$. Hence, $z > 0$ and therefore $u_3^{\text{new}} \geq 0$. This gives rise and justifies the AOPT algorithm (see Algor. 2.1).

Algorithm 2.1 The AOPT algorithm.

for $m = 1, \ldots, M$: **do**
  Start from a complementary partition (2.13), which has the property (2.14).
  Compute $\bar{u}_{n_b}^{\text{old}}$.
  if $n_b = 0$ **then**
    stop
  end if.
  while $s_{n_b}^{\text{old}} < 0$, **do**
    $n_b := n_b - 1$.
    Compute $\bar{u}_{n_b}^{\text{new}}$ according to (2.15-2.16).
    $\bar{u}_{n_b}^{\text{old}} := \bar{u}_{n_b}^{\text{new}}$.
    if $n_b = 0$ **then**
      stop
    end if.
  end while
end for

Note the following implementation detail: the quantity $A_{33}^{-1} e_1$ entering the new basic solution can be implemented by updates similarly to the updating scheme of (2.15-2.16). By defining $x^{\text{old}} := \gamma A_{33}^{-1} e_1$ one can update x^{new} according to:

$$x^{\text{new}} = \begin{pmatrix} y \\ x^{\text{old}}\, y \end{pmatrix},$$

where

$$y = \frac{\gamma}{\alpha - \beta e_1^T x^{\text{old}}} \quad \blacksquare$$

Corollary. *From the above proof it is clear that the "While do loop" of the algorithm* AOPT *completes in at most n steps.* (The proof is evident by the AOPT algorithm and the Theorem.)

Now we continue with the proof of the Lemma.

Proof of the Lemma. To prove the property (2.14) we first prove that:

The right hand side of the first time step has the following sign structure:

$$b^o = \begin{pmatrix} - \\ \oplus \end{pmatrix}. \tag{2.17}$$

Indeed, since $u^o = 0$ we have:

$$b^o \equiv b = \phi - (A+B)\psi,$$

or by its elements:

$$b_i = -\rho[\psi_i - \tilde{q}\psi_{i+1} - (1-\tilde{q})\psi_{i-1}] - r\Delta t\psi_i, \quad i = 1,\ldots,n.$$

Assume that i_o is the index such that:

$$\psi_{i\le i_o} = K - e^{\xi_i} > 0, \quad \psi_{i>i_o} = 0.$$

Then one has

$$b_{i_o+1} = \rho(1-\tilde{q})\psi_{i_o} \ge 0,$$

and therefore $b_{i>i_o} \ge 0$. If i_o is such that $K = e^{\xi_{i_o}}$ then $b_{i\ge i_o} \ge 0$, otherwise b_{i_o} may have either sign.

To analyze the $i < i_o$ case one may define a new vector c_i by:

$$b_i = \rho c_i - r\Delta t\psi_i,$$

where

$$c_i = \tilde{q}\psi_{i+1} + (1-\tilde{q})\psi_{i-1} - \psi_i,$$

or

$$c_i = e^{\xi_i}(1-\cosh\Delta\xi) + e^{\xi_i}(1-2\tilde{q})\sinh\Delta\xi.$$

The first term is always negative, whereas the second term may be of either sign: for $\tilde{q} \ge \frac{1}{2}$ it is non-positive and therefore $b_{i<i_o} < 0$ too. In case of $\tilde{q} < \frac{1}{2}$ the terms are competing and the sign of $b_{i<i_o}$ may be positive for some indices i near i_o. In any case one can restrict $\Delta\xi$ to be small enough such that $b_{i<i_o} < 0$. For example, a sufficient condition may be the non-positivity of $c_{i<i_o}$. Using the definition of $\tilde{q}$ and a simple algebra yields:

$$\frac{\frac{\Delta\xi}{2}}{\tanh\frac{\Delta\xi}{2}} \le \frac{1}{1-\frac{2r}{\sigma^2}}. \tag{2.18}$$

This condition implies that in the extreme case of high volatility $(\Delta\xi)^2$ should of the order $O(r/\sigma^2)$.

A consequence of the above structure is the property (2.14) for the first time step $m = 1$. Indeed, since $A^{-1} \geq 0$ by virtue of (2.10b) we can make a partition (2.13) with the property (2.14) by taking as b_3 the nonnegative part of the starting right hand side b^o.

Finally let us prove the Lemma for any other time step. The proof is by induction.

For $m = 1$ the Lemma is true. Let us show it holds also for $m = 2$. Using (2.9) for $\theta = 1$ we get:

$$b^2 = b^1 + u^1, \tag{2.19}$$

where u^1 is the solution to the LCPs (2.8) for $m = 1$. Since $A_{33}u_3^1 = b_3^1$ we have:

$$A_{33}^{-1}b_3^2 = u_3^1 + A_{33}^{-1}u_3^1 \geq 0. \tag{2.20}$$

From (2.19) we have $b_1^2, b_2^2 < 0$ and the Lemma is proven for $m = 2$.

From the Algorithm AOPT it is clear that the feasible basic solution is updated by positive increments. Therefore one has:

$$u_3^2 \geq A_{33}^{-1}b_3^2, \tag{2.21}$$

from which it follows that:

$$u^2 \geq u^1. \tag{2.22}$$

Until now together with the Lemma for $m = 1,2$ we have proven the following *monotonicity*:

$$u^m \geq u^{m-1}, \quad m = 1,2. \tag{2.23}$$

Finally let us assume that the Lemma and the above monotonicity are true at the time slice $m-1$ and show that they hold at the time slice m. Again using (2.9) for $\theta = 1$ we get:

$$b^m = b^{m-1} + u^{m-1} - u^{m-2}, \tag{2.24}$$

and since $A_{33}^{-1}u_3^{m-1} = b_3^{m-1}$ we have:

$$A_{33}^{-1}b_3^m = u_3^{m-1} + A_{33}^{-1}(u_3^{m-1} - u_3^{m-2}). \tag{2.25}$$

Using the monotonicity at $m-1$, i.e. $u_3^{m-1} \geq u_3^{m-2}$ it follows that $A_{33}^{-1}b_3^m \geq 0$. Applying the AOPT Algorithm we solve the LCP for this time slice. Since the feasible basic solution is updated by positive increments we have:

$$u_3^m \geq A_{33}^{-1}b_3^m. \tag{2.26}$$

Using (2.25) we get:

$$u^m \geq u^{m-1}, \tag{2.27}$$

i.e. the monotonicity is proven also for this time slice. From (2.24) follows that $b_1^m, b_2^m < 0$ and the Lemma is proven for this time slice. ■

Remark. We have shown that the right hand side for the 1^{st} time slice has a certain sign structure for $\Delta\xi$ satisfying the inequality (2.18). In fact, for a zero risk-free interest rate r this condition it is not fulfilled. As a consequence the 1^{st} time slice right hand side is nonnegative and there are no LCPs to be solved. In this case the option is optimally held to maturity.

From Lemma the starting complementary partition at each time step may be given by the last complementary partition of the previous time step. This simplifies the AOPT algorithm which is summarized by Algor. 2.2, where all partitions of the form (2.13) correspond to the complementary basic solutions of the form (2.12).

Algorithm 2.2 The AOPT1 algorithm.

Take n_b such that $b_1^o, b_2^o < 0$ and $b_3^o \geq 0$.
for $m = 1, \ldots, M$: **do**
  $y = \gamma\, e_1^T A_{33}^{-1} e_1,\ z = e_1^T A_{33}^{-1} b_3$.
  if $n_b = 0$, **then**
    $s_2 := 0$,
  else
    $s_2 = -(b_2 + \beta z)$.
  end if
  while $s_2 < 0$, **do**
    $n_b := n_b - 1$.
    $z = -s_2/(\alpha - \beta y)$.
    if $n_b = 0$, **then**
      $s_2 := 0$,
    else
      $s_2 := -(b_2 + \beta z),\ y := \gamma/(\alpha - \beta y)$.
    end if
  end while
  $u_3^m = A_{33}^{-1} b_3,\ u_1^m := 0, u_2^m := 0$.
  $b := b^o - Bu^m$.
end for

Since at each time step the algorithm AOPT1 calls only three times a linear system solver, we conclude that it has an $O(n)$ numerical complexity. (It takes an order $O(n)$ operations to compute the LU decomposition of the tridiagonal matrix A [Golub and Van Loan, 1989].) Hence the overall complexity of the algorithm AOPT1 is $O(n \times M)$. In Fig. 2.1 we show as an example the efficient frontier of an American option computed using the algorithm AOPT1.

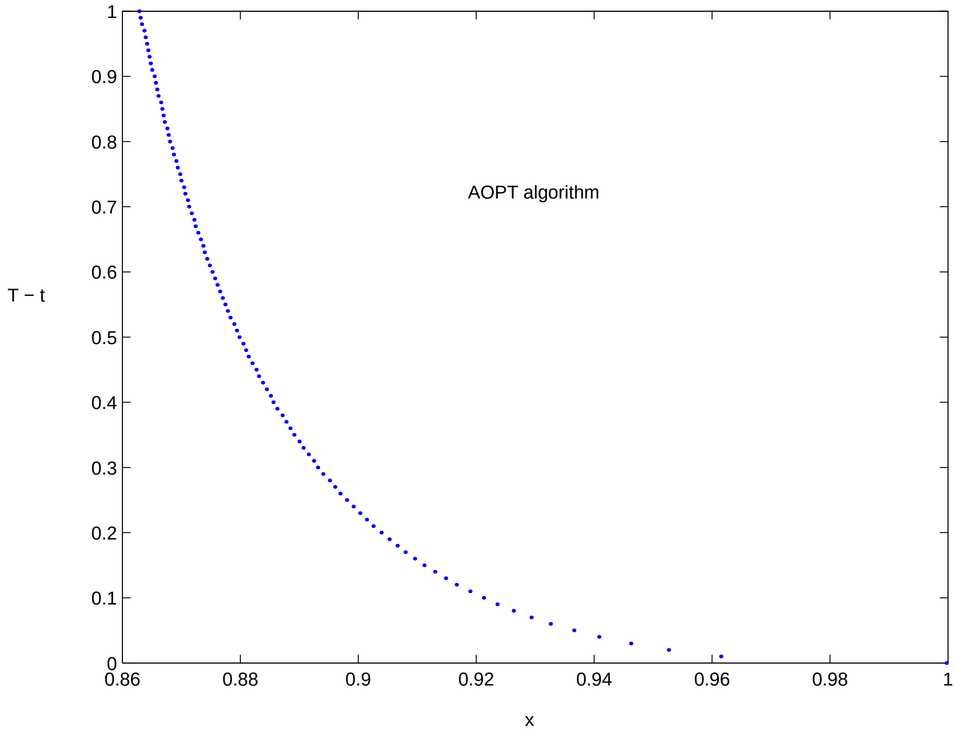

Figure 2.1. Efficient frontier of an American put option computed by the AOPT1 algorithm described in the text and $\theta = 1$ for 100 time steps and $\rho = 6400$. The option parameters are: interest rate $r = 0.1$, volatility $\sigma = 0.2$, strike price $K = 1$ and maturity $T = 1$.

4. Concluding remarks

The main result of this work is that the American options can be evaluated in linear time using the algorithm AOPT1. On the other hand such an application represents a class of structural LCPs that can be solved in linear time.

We note that although the proof relies on the implicit difference scheme, our experience with the algorithm AOPT1 is that it works for all $\theta \in [\frac{1}{2}, 1]$, $\theta = \frac{1}{2}$ (Crank-Nicholson scheme) being the most accurate: it's time discretization errors are of the order $O((\Delta t)^2)$ [Tavella and Randall, 2000].

However, for $\theta = \frac{1}{2}$ the computed option prices do not satisfy the monotonicity property (2.27). In this case, for large ρ values the eigenvectors corresponding to high eigenvalues of the discrete Black-Scholes operator become apparent in the discrete value function. Such components are artifacts of the discretization and their presence should be avoided. (see also [Tavella and Randall, 2000])

This hints that although popular in finance, the Crank-Nicholson scheme is not robust. One must work with moderate ρ or with small time steps to achieve the desired properties of the option's value function [Myneni, 1992].

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