Convex Set Operators and Polynomial Integer Minimization in Fixed Dimension

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Joint work with
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IBM ETH ETH
Polynomial IP
Consider the problem

\[
\min \{ f(x) : x \in P \cap \mathbb{Z}^n \},
\]

(1)

where \( P \) is the rational polyhedron \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) with \( A \in \mathbb{Z}^{m \times n}, \ b \in \mathbb{Z}^m, \ f \in \mathbb{Z}[x] \) has degree \( d \).

- **Complexity:** We want time-complexity when input is binary encoding of \( A, b \) and the coefficients of \( f \).
- **Fixed:** Dimension \( n \), degree \( d \) of \( f \).

$$\min \quad (x^2 - ay - c)^2$$
$$1 \leq x \leq c - 1,$$
$$\frac{1-a}{b} \leq y \leq \frac{(c-1)^2-a}{b},$$
$$x, y \in \mathbb{Z}.$$

Implies AN1 problem - given three positive integers $a, b, c$, determine if there exist $x \in \mathbb{Z}$ such that $x^2 \equiv a \pmod{b}$ with $x < c$. 

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\]

Implies AN1 problem - given three positive integers $a, b, c$, determine if there exist $x \in \mathbb{Z}$ such that $x^2 \equiv a \pmod{b}$ with $x < c$.

2. [Del-Pia and Weismantel 2014] **Polynomial time** for quadratic polynomials in 2 variables.
Other complexity results

Negative

1. [Matiyasevich 1977, Jones 1982 (Hilbert’s 10th problem, 1900)]
   Undecidable in 58 variables with degree 4.

Positive

1. FPTAS for non-negative maximization in fixed dimension (DeLoera, Hemmeke, Köppe, Weismantel 2011)
2. NP Integer quadratic programming (Del Pia, Dey, Molinaro 2014)
$f$ is a polynomial of degree $d$,

$$\min\{f(x) : x \in P \cap \mathbb{Z}^n\},$$

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Main Results

Theorem (Del Pia, H., Weismantel, Zemmer)

The problem \( \min \{ f(x) : x \in P \cap \mathbb{Z}^2 \} \) is

1. polynomial time when \( f \) is cubic,
2. polynomial time when \( f \) is homogeneous, fixed degree, \( P \) is bounded,
3. "Intractable" when \( f \) is homogeneous, degree 4, \( P \) unbounded.
Main Results

Theorem (Del Pia, H., Weismantel, Zemmer)

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1. polynomial time when $f$ is cubic,
2. polynomial time when $f$ is homogeneous, fixed degree, $P$ is bounded,
3. "Intractable" when $f$ is homogeneous, degree 4, $P$ unbounded.

"Intractable" := Requires compact representation.

$$\min\{(x^2 - Ny^2)^2 : (x, y) \in \mathbb{Z}_{\geq 1}^2\}, \quad N = 5^{2k+1}$$

- Optimal objective value is 1.
- Minimal size solution satisfies Negative Pell Equation $x^2 - Ny^2 = -1$.
- Lagarias (1980) - minimal solution has binary encoding size $\Omega(5^k)$. 
Let $C \subseteq \mathbb{R}^n$ be a convex set.

- A function $f : C \rightarrow \mathbb{R}$ is quasi-convex on $C$ if $\{x \in C : f(x) \leq \alpha\}$ is convex for all $\alpha \in \mathbb{R}$.

**Lemma (Convex Lemma (Khachiyan-Porkolab 2000))**

1. $C$ is a bounded convex semi-algebraic set,
2. $f : \mathbb{R}^n \rightarrow \mathbb{R}$, polynomial, quasi-convex on $C$.

In polynomial time (fixed dimension), we can solve the problem

$$\min \{f(x) : x \in C \cap \mathbb{Z}^n\}.$$
Let $C \subseteq \mathbb{R}^n$ be a convex set.

- A function $f : C \to \mathbb{R}$ is quasi-concave on $C$ if $\{x \in C : f(x) \geq \alpha\}$ is convex for all $\alpha \in \mathbb{R}$.

**Lemma (Concave Lemma)**

1. $P$ a bounded polyhedron,
2. $f : P \to \mathbb{R}$, quasi-concave on $P$.

In polynomial time (fixed dimension), we can solve the problem

$$\min\{f(x) : x \in P \cap \mathbb{Z}^n\}.$$

**Proof.**

Let $x^*$ be an optimal vertex. Then

$$P_I = \text{conv}(\text{vert}(P_I)) \subseteq \text{conv}(\{x : f(x) \geq f(x^*)\}) = \{x : f(x) \geq f(x^*)\}$$

Enumerate $\text{vert}(P_I)$ with Cook-Kannan-Hartman-McDiarmid (1992).
Quadratic Minimization Review

Del Pia-Weismantel (2014) - quadratic solved in polynomial time.

\[ f(x, y) \approx x^2 + y^2 \]

**Convex:** Apply Khachiyan-Porkolab 2010 test feasibility + binary search.

\[ f(x, y) \approx -x^2 - y^2 \]

**Concave:** implies solution on vertex of integer hull \( P_I \).


\[ f(x, y) \approx x^2 - y^2 \]

**Divide problem** into quasi-convex and quasi-concave regions.
Homogeneous Case

\[ f(x) \text{ homogeneous if} \]
\[ f(\lambda x) = \lambda^d f(x) \text{ for all } \lambda \in \mathbb{R}. \]

\[ f(x) \text{ homogeneous polynomial if} \]
\[ \text{every monomial has the same degree, i.e., } f(x, y) = x^5 + 3y^2x^3. \]

Quasi-convex/Quasi-concave division

- Zeros occur on lines.
- Function no longer quasi-concave/quasi-convex in positive/negative regions.
- Domain can still be divided into quasi-concave/quasi-convex regions.
- Numerically approximate regions.
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$(x + y)(x - y)(x^2 + y^2)^4$
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Bordered Hessian, Homogeneous, Volume Argument

1. Bordered Hessian

\[ D_h = \det \left[ \begin{array}{cc} 0 & \nabla h^T \\ \nabla h & \nabla^2 h \end{array} \right] \text{ hom.} \equiv -\frac{d}{d-1} h(x) \cdot \det(\nabla^2 h(x)) \]

2. \( D_h < 0 \Rightarrow \) quasi-convex while \( D_h > 0 \Rightarrow \) quasi-concave (\( \mathbb{R}^2 \) only).

3. \( D_h \equiv 0 \iff h(x) = (c^T x)^d \) (Hemmer, 1995)

4. \( D_h \) is a homogeneous polynomial \( \Rightarrow \) zeros occur on lines.
**Bordered Hessian, Homogeneous, Volume Argument**

1. **Bordered Hessian**

   \[ D_h = \det \begin{bmatrix} 0 & \nabla h^T \\ \nabla h & \nabla^2 h \end{bmatrix} \overset{\text{hom.}}{=} \frac{-d}{d-1} h(x) \cdot \det(\nabla^2 h(x)) \]

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1. Compute zeros of \( D_h(x_1, \pm R) \) to appropriate accuracy

2. Create boxes containing zero lines

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**Lemma (BOWW 2013)**

*If \( C \subseteq \mathbb{R}^2 \) is convex, \( \text{vol}(C) < 1/2 \), then \( \dim(C \cap \mathbb{Z}^2) \leq 1 \).*
Homogeneous Minimization

Theorem - Quasiconvex/Quasiconcave division

Let \( f \) be a homogeneous translatable polynomial. In polynomial time we can find a polynomial number of rational polyhedra \( P_i, Q_j \) and rational lines \( L_k \) such that

- \( f \) is quasi-convex on \( P_i \),
- \( f \) is quasi-concave on \( Q_j \), and

\[
P \cap \mathbb{Z}^2 = \left( \bigcup_{i=1}^{\ell_1} P_i \cup \bigcup_{j=1}^{\ell_2} Q_j \cup \bigcup_{k=1}^{\ell_3} L_k \right) \cap \mathbb{Z}^2.
\] (2)

Theorem - Homogeneous, Bounded

In polynomial time, we can minimize a homogeneous translatable polynomial of fixed degree in 2 variables over the integer points of a bounded polyhedron.
Let $C$ be a convex semi-algebraic set. In fixed dimension, we can determine in polynomial time if the following sets are non-empty:

1. $\left( P \cap C \right) \cap \mathbb{Z}^n$ [KP, 2000]
2. $\left( P \setminus C \right) \cap \mathbb{Z}^n$ Enumerate vertices of $P_I$ [CKHM, 1993]
Tools #2: Feasibility Using Convex Set Operator

Theorem: Convex Set Operator

Let $C$ be a convex semi-algebraic set. In fixed dimension, we can determine in polynomial time if the following sets are non-empty:

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   “Quasiconvex"

2. $(P \setminus C) \cap \mathbb{Z}^n$  
   Enumerate vertices of $P_I$  
   [CKHM, 1993]  
   “Quasiconcave"

**Definition**

A *division description* of $S_{\leq \omega}^f$ on $P$ is a list of polyhedra $P_i, Q_j$ and lines $L_k$ that cover $P \cap \mathbb{Z}^2$ such that

1. $C := P_i \cap S_{\leq \omega}^f$ is convex

2. $C := Q_j \cap S_{> \omega}^f$ is convex
Tools #2: Division description
$P_1, P_2, Q_1, Q_2, L_1$ is a division description of $S_{\leq \omega}^f$.

1. $P_1 \cap S_{\leq \omega}^f$, $P_2 \cap S_{\leq \omega}^f$ are convex.
2. $Q_1 \cap S_{> \omega}^f$, $Q_2 \cap S_{> \omega}^f$ are convex.
3. $P \cap \mathbb{Z}^2 \subseteq P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup L_1$
Tools #2: Division description

\[ P_1, P_2, Q_1, Q_2, L_1 \] is a division description of \( \mathcal{S}_{\leq \omega} \).

1. \( P_1 \cap \mathcal{S}_{\leq \omega}, P_2 \cap \mathcal{S}_{\leq \omega} \) are convex.

2. \( Q_1 \cap \mathcal{S}_{> \omega}, Q_2 \cap \mathcal{S}_{> \omega} \) are convex.

3. \( P \cap \mathbb{Z}^2 \subseteq P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup L_1 \)

Theorem: Division \( \Rightarrow \) Optimization

Suppose \( P \) is bounded and that for every \( \omega \in \mathbb{Z} + \frac{1}{2} \) we can compute a division description of \( \mathcal{S}_{\leq \omega} \) in polynomial time. Then, we can solve \( \min \{ f(x) : x \in P \cap \mathbb{Z}^n \} \) in polynomial time.
Cubic Minimization
Critically Affine

Definition
A function is said to be critically affine if its gradient vanishes exactly on a finite union of affine spaces.

Theorem
Suppose $f(x, y)$ is one of the following:

1. Cubic polynomial,
2. Homogeneous polynomial,
3. Separable polynomial.

Then $f(x, y)$ is critically affine.

NP-hard example

\[(x^2 - ay + c)^2\]

Gradient vanishes on the parabola
\[ f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3 \]

- **Constant-Cubic**
  \[ f(x, y) = f_0(x) \]

Then the set \( \{(x, y) : f(x, y) \leq \omega\} \) is union of cylinders.

\[ \Rightarrow \text{Division description is easy.} \]
\[ f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3 \]

- **Constant-Cubic**

\[ f(x, y) = f_0(x) \]

Then the set \( \{(x, y) : f(x, y) \leq \omega\} \) is union of cylinders.

\[ \Rightarrow \] Division description is easy.

- **Linear-Cubic**

\[ f(x, y) = f_0(x) + f_1(x)y \]

Curve \( f(x, y) = \omega \), described by \( y = g(x) := (\omega - f_0(x))/f_1(x) \)

Find inflection points and vertical asymptotes of \( g(x) \) and divide into vertical cylinders about these points.
• Quadratic-Cubic

\[ f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 \]

Then \( f(x, y) = \omega \) on curves

\[ y_{\pm} = \frac{-f_1 \pm \sqrt{f_1^2 - 4(f_0 - \omega)f_2}}{2(f_0 - \omega)} \]

1. Compute vertical asymptotes
2. Compute intersections of \( y_+, y_- \)
3. Compute inflection points (changes in concavity) of \( y_+, y_- \)
**Quadratic-Cubic: Division Description**

- **Quadratic-Cubic**

\[
\begin{align*}
f(x, y) &= f_0(x) + f_1(x)y + f_2(x)y^2
\end{align*}
\]

Then \( f(x, y) = \omega \) on curves

\[
y \pm \sqrt{\frac{f_1^2 - 4(f_0 - \omega)f_2}{2(f_0 - \omega)}}
\]

1. Compute vertical asymptotes
2. Compute intersections of \( y_+, y_- \)
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4. Numerical approximations
Quadratic-Cubic: Division Description

- Quadratic-Cubic

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1. Compute vertical asymptotes
2. Compute intersections of \( y_+, y_- \)
3. Compute inflection points (changes in concavity) of \( y_+, y_- \)
4. Numerical approximations
**Lemma**

Any line can intersect a cubic level set at most 3 times.

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**NP-hard Quartic**

\[(x^2 - ay + c)^2\]

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**Cubic Polynomials**

- Concave/Concave
- Concave/Convex
- Convex/Concave

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Robert Hildebrand (ETH)
Cubic-Cubic: Approximate Rotation to Quadratic-cubic

\[ x^3 + y^3 + 9 \]

\[ 3u^2v - 3uv^2 + v^3 + 9 \]

\[ x^3 + \delta x^2y + y^3 + 8.96 \]

\[ 0.05u^3 + 2.89u^2v - 2.95uv^2 + v^3 + 8.97 \]

**Lemma**

\[ \{(x, y) \in P \cap \mathbb{Z}^2 : f(x, y) \leq \omega + \frac{1}{2}\} = \{(x, y) \in P \cap \mathbb{Z}^2 : f_\epsilon(x, y) \leq \omega + \frac{1}{2}\} \]
Cubic-Cubic: Approximate Rotation to Quadratic-cubic

\[ x^3 + y^3 + 9 \]

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Lemma

\[ \{(x, y) \in P \cap \mathbb{Z}^2 : f(x, y) \leq \omega + \frac{1}{2}\} = \{(x, y) \in P \cap \mathbb{Z}^2 : f_\varepsilon(x, y) \leq \omega + \frac{1}{2}\} \]
Cubic Minimization (Bounded)

Theorem - Cubic Division Description

Let $f$ be a cubic polynomial in 2 variables with integer coefficients and let $\omega \in \mathbb{Z} + \frac{1}{2}$. In polynomial time, we can determine a division description of $S^f_{\leq \omega}$ on $P$.

Theorem - Cubic, Bounded

In polynomial time, we can minimize a cubic polynomial in 2 variables over the integer points of a bounded polyhedron.
Cubic Minimization (Unbounded)
Theorem: Cubic minimization unbounded

In polynomial time, we can either decide that the problem is unbounded, or find a polynomial size bound on the feasible region that must contain the optimal solution.

Case analysis:

\[
f(x, y) = \underbrace{(x^3 + xy^2)}_{h(x,y)} + \underbrace{(x^2 + xy - 3x + 2)}_{g(x,y)}
\]

Assume \( P \cap \mathbb{Z}^2 \neq \emptyset \) and \( \text{rec}(P) \neq \emptyset \) is pointed.

1. \( h(r) < 0 \) for some \( r \in \text{rec}(P) \)

Then

\[
f(x + \lambda r) = \lambda^3 h(r) + O(\lambda^2)\bar{g}(x, r) \approx \lambda^3 h(r) \to -\infty
\]

Problem is unbounded.
**Case Analysis:**

\[ f(x, y) = \underbrace{x^3 + xy^2}_{h(x,y)} + \underbrace{x^2 + xy - 3x + 2}_{g(x,y)} \]

Assume \( P \cap \mathbb{Z}^2 \neq \emptyset \) and \( \text{rec}(P) \neq \emptyset \) is pointed.

1. \( h(r) > 0 \) for all \( r \in \text{rec}(P) \)
   - Find lower bound \( m \leq h(r) \) for all \( r \in \text{rec}(P) \) with \( ||r|| = 1 \).
   - Use Rouche's Theorem
   - Find decomposition \( P = Q + \text{rec}(P) \), \( Q \) bounded, \( q \in Q \)

\[
  f(q + \lambda r) = \lambda^3 h(r) + O(\lambda^2) \bar{g}(q, r) \geq \lambda^3 m + O(\lambda^2) \bar{g}(q, r) \geq f(x)
\]

for all \( \lambda \geq M \).

- Problem is bounded by \( R := R_Q + M \)
Case analysis: \( f(x, y) = (x^3 + xy^2) + (x^2 + xy - 3x + 2) \)

Assume \( P \cap \mathbb{Z}^2 \neq \emptyset \) and \( \text{rec}(P) \neq \emptyset \) is pointed.

1. \( h(r) > 0 \) for all \( r \in \text{rec}(P) \)
   
   - Find lower bound \( m \leq h(r) \) for all \( r \in \text{rec}(P) \) with \( ||r|| = 1 \).
     
     ⇒ Use Rouche’s Theorem
   
   - Find decomposition \( P = Q + \text{rec}(P) \), \( Q \) bounded, \( q \in Q \)
   
   \[
   f(q + \lambda r) = \lambda^3 h(r) + O(\lambda^2)\bar{g}(q, r) \geq \lambda^3 m + O(\lambda^2)\bar{g}(q, r) \geq f(x)
   \]
   
   for all \( \lambda \geq M \).

2. Problem is bounded by \( R := R_Q + M \)

3. \( h(r) \geq 0 \) for all \( r \in \text{rec}(P) \)
   
   .... Need to analyze \( r \) where \( h(r) = 0 \).
Homogeneous cubic polynomials

\[ h(x, y) = \prod_{i=1}^{3} (a_i x + b_i y) \]

\[ h(x, y) = (a_1 x + b_1 y)^2 (a_2 x + b_2 y) \]

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Case 3a: \( h \geq 0, \ h(r) = 0 \)

**Lemma: Rational Zeros**

If \( h(\tilde{r}_1, \tilde{r}_2) = 0 \), and \( h \geq 0 \) on all of \( \text{rec}(P) \), then \( \tilde{r}_1/\tilde{r}_2 \) is rational and it can be computed in polynomial time.
Case 3a: $h \geq 0, \ h(r) = 0$

**Lemma: Rational Zeros**

If $h(\bar{r}_1, \bar{r}_2) = 0$, and $h \geq 0$ on all of $\text{rec}(P)$, then $\bar{r}_1/\bar{r}_2$ is rational and it can be computed in polynomial time.

Fix $\bar{r} \in \text{rec}(P)$ with $h(\bar{r}) = 0$. 
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$$f(x + \lambda \bar{r}) = \lambda^2 g_{\bar{r},2}(x) + \lambda g_{\bar{r},1}(x) + g_{\bar{r},0}(x).$$
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Note that

- $f(x + \mu \bar{r} + \lambda \bar{r}) = \lambda^2 g_{\bar{r},2}(x + \mu \bar{r}) + L.O.T.$
- $f(x + (\mu + \lambda) \bar{r}) = (\lambda + \mu)^2 g_{\bar{r},2}(x) + L.O.T.$

Matching terms on $\lambda^2$, we see that $g_{\bar{r},2}(x + \mu \bar{r}) = g_{\bar{r},2}(x)$.

$\Rightarrow g_{\bar{r},2}(x) = \bar{g}(x \cdot \bar{r}^\perp).$
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$$\Rightarrow g_{\bar{r},2}(x) = \bar{g}(x \cdot \bar{r}^\perp).$$

1. Handle separately $g_{\bar{r},2}(x) = 0$ on these lower dimensional slices.
2. If $\exists x$ with $g_{\bar{r},2}(x) < 0$, then problem is unbounded.
3. If all $x$ satisfy $g_{\bar{r},2}(x) > 0$, then problem is bounded.

- If $g_{\bar{r},2}(x) \equiv 0$, then proceed with $g_{\bar{r},1}(x)$. 
**Case 3b:** \( h \geq 0, \ h(r) = 0, \ g_{r,2}(x) \equiv 0 \)

- Suppose \( g_{r1,1} \neq 0 \).
- Show that \( g_{r1,1}(x + \mu r^1) = g_{1}^r(x) \).
- Compute \( g^* = \min\{g_{r1,1}(x) : x \in P_1 \cap \mathbb{Z}^2\} \)

1. Handle separately \( g^* = 0 \) on these lower dimensional slices.
2. If \( g^* > 0 \), then problem is **unbounded**.
3. If \( g^* < 0 \), then problem is **bounded**.

- If \( g_{r1,1}(x) \equiv 0 \), then proceed to easy Case 3c that can be solved easily since it follows that \( f(x) = f(x + \lambda r^1) \) for all \( \lambda \in \mathbb{R} \).
Theorem (Del Pia, H., Weismantel, Zemmer)

The problem \( \min \{ f(x) : x \in P \cap \mathbb{Z}^2 \} \) is

1. polynomial time when \( f \) is cubic,
2. polynomial time when \( f \) is homogeneous, fixed degree, \( P \) is bounded,
3. "Intractable" when \( f \) is homogeneous, degree 4, \( P \) unbounded.
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- Techniques apply to low rank minimization problem

\[
\min \{ f(x, y) : (x, y, z) \in P \cap \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^n \}
\]
### Open Questions

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How to do get into 3-dimensions and higher?

Robert Hildebrand (ETH)
### Open Questions

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Thank you for listening!