Valuing virtual production capacities on flow commodities

Juri Hinz
Institute for Operations Research and RiskLab
ETH Zentrum
CH–8092 Zurich, Switzerland
e-mail hinz@ifor.math.ethz.ch

June 15, 2006

Abstract

As a result of storability restrictions, the price risk management of flow commodities (such as natural gas, oil, and electrical power) is by no means a trivial matter.To protect price spikes, consumers purchase diverse swing–type contracts, whereas contract writers try to hedge themselves by appropriate physical assets, for instance, using storage utilities, through transmission and/or production capacities. However, the correct valuation of such contacts and their physical counterparts is still under lively debate. In this approach, an axiomatic setting to discuss price dynamics for flow commodity contracts is suggested. By means of a minimal set of reasonable assumptions we suggest a framework where the standard change–of–numeraire transformation converts a flow commodity market into a market consisting of zero bonds and some additional risky asset. Utilizing this structure, we apply the toolkit of interest rate theory to price the availability of production capacity on a flow commodity.

Key words: swing option, electricity risk, energy economics, futures markets, power derivatives

1 Introduction

In the following, electricity pricing will be studied using electrical power as the primary flow commodity example in this contribution.

In practice, diverse technical problems prevent electricity from being traded as a conventional commodity. One of the most complicated of these problems is the *balancing restriction*, which requires that the demand and the supply of electricity be equal at any time. As a result, the electricity market consists of two segments: one for contract trading on immediate energy production (so–called balancing market, usually organized following an auction-like principle) and one on future delivery (effected at futures market, by conventional trading). Within the latter, a remarkable trading activity can be observed for hourly contracts with delivery within the following day (the so-called *day-ahead* market). As a rule, positions taken here imply physical energy delivery/consumption, which explains why electricity day-ahead prices are also referred to as *spot* prices. An *electricity retailer* is obliged to cover the random demand of its end consumers at a fixed price. Doing so, retailers face high risk: in the case that consumer's demand tops the own supply, the missing energy is purchased at the balancing market price, which could be markedly high. One rational way of handling this risk is to accurately predict the demand, in order to adjust the day-ahead position as precisely as possible. In practice, short-term demand forecast seems to be a minor problem; apparently, an efficient protection against undesirable long–term spot price movements turns out to be the more important issue. Here, long term options written on spot prices are popular. However, due to difficulties in valuation and hedging of electricity derivatives, option writers prefer to sell agreements which are, at least approximately, replicated by appropriate physical assets. As a result, it can be observed that many electricity derivatives are of a swing type, presenting corresponding financial counterparts of agreements on production capacities. As an example, let us discuss the *virtual production capacity.*

Consider a power plant where the owner decides for any period (for example, a day) $[t_k, t_{k+1}]$ within a given time interval $[0, \vartheta] = \cup_{k=0}^{N-1}[t_k, t_{k+1}]$ to produce electrical power at intensity q_{t_k} subject to technical constraints

$$(1) \qquad 0 \leq q_{t_k} \leq \lambda \text{ for all } k = 0, \dots, N-1, \qquad \sum_{k=0}^{N-1} q_{t_k}(t_{k+1} - t_k) \leq \Lambda$$

with given maximal electrical power $\lambda > 0$ (MW) and the total amount of energy $\Lambda > 0$ (MWh). The electricity scheduled for production for the day $[t_k, t_{k+1}[$, is sold on the day–ahead market at the spot price E_{t_k}, giving the total revenue

$$\sum_{k=0}^{N-1} q_{t_k}(E_{t_k} - K)^+(t_{k+1} - t_k) \tag{2}$$

where K stands for production costs. Note that here we have assumed that the payoff for the day $[t_k, t_{k+1}[$ is proportional to $(E_{t_k} - K)^+$ meaning that production runs only if the price covers the production costs. This assumption is unrealistic for scheduling the plant at a higher time resolution (say, for half–hourly adjusted dispatch on the balancing market) since we can not assume that production would stop immediately when the marginal revenue became negative. However, when dispatching the plant on the basis of the day–ahead market, revenue (2) subject to restrictions (1) gives a realistic model. In what follows, we deal with a continuous–time reformulation of the above framework. More precisely, we approximate dispatch policies with (1) by continuous–time processes $(q_t)_{t\in[0,T]}$ satisfying

$$0 \le q_t \le \lambda \text{ for all } t \in [0, \vartheta], \qquad \int_0^{\vartheta} q_t dt \le \Lambda \tag{3}$$

whereas the total revenue (2) is replaced by

$$\int_0^{\vartheta} q_t(E_t - K)^+ dt \tag{4}$$

where $(E_t)_{t\in[0,\vartheta]}$ denotes a continuous–time analog for the electricity spot price process. Due to such reformulation, results available for diffusion processes from financial mathematics and optimal control theory can be utilized.

Suppose now that the ideal power plant described by (3) and (4) is equivalently transformed to a financial agreement, referred to as *virtual production capacity* in the sequel. Denote by $R_t := (E_t - K)^+$ the revenue intensity at time $t \in [0, \vartheta]$. The holder of a virtual production capacity can opt any exercise policy $(q_t)_{t\in[0,\vartheta]}$ obeying (3), whereas the contract writer is obliged to supply a cash–flow at intensity $(q_t R_t)_{t\in[0,\vartheta]}$ depending on the opposite party's exercise policy $(q_t)_{t\in[0,\vartheta]}$. Following an exercise policy $(q_s)_{s\in[0,\vartheta]}$ the agent possesses at time t the capacity level $\rho_t^q := \Lambda - \int_0^t q_s ds$. If the holder decides at this time to sell the virtual production capacity to another market participant, then for the remaining time $]t, \vartheta]$ the new owner is given the right to exercise the contract by any policy $(u_s)_{s \in]t, \vartheta]}$, subject to restrictions

$$0 \leq u_s \leq \lambda, \quad s \in]t, \vartheta], \qquad \int_t^{\vartheta} u_s ds \leq \rho_t^q.$$

to receive a cash flow at intensity $(u_s R_s)_{s \in]t, \vartheta]}$. In the sequel, we discuss the valuation of virtual production capacity to show that under certain conditions, the fair initial price of this contract is given by

$$\sup_q E_Q \left(\int_0^{\vartheta} q_t R_t / B_t dt \right) \tag{5}$$

where $(B_t)_{t \in [0, \vartheta]}$ denotes the value of a savings account and Q is some risk–neutral measure. The supremum in (5) arises over a set of policies satisfying (3). Whereas this supremum–form is obtained in Section 4 utilizing arguments from American put valuation (see [15]), the major part of our approach deals with the concept of risk–neutral spot price dynamics. The difficulty here is that flow commodity spot prices (for example, electricity spot prices) at different times are not directly related to each other, strictly speaking, E_s and E_t are to be considered as prices for different commodities delivered on different dates $s \neq t$. In Section 2, flow commodity price models satisfying a minimal set of reasonable axioms are studied: (i) the price evolution is described by stochastic processes with appropriate path properties, (ii) the model explains the initial yield curve, (iii) it excludes arbitrage opportunities, and (iv) it reflects storability restrictions. We make precise these requirements and show how to construct stochastic models satisfying them.

Some related research in this field must also be mentioned. The connection between spot and forward prices for commodities with restricted storability and valuation of storage opportunities has attracted research interest for a long time. We emphasize here, among others, the work [4], [6] as well as [11], [21] and a general model in [18]. Related to this work, the authors of [8] expose questions of electricity pricing and explain that the non–storability issue requires a production process model. Another research direction (see [1], [13], [17], [7], [12]) focuses on modeling the stochastic process of spot price, where the three last contributions also develop a risk-neutral point view on the electricity spot price process. Moreover, a valuation method for electricity swing options has been considered in [7].

Finally, the review paper [5] provides a valuable overview on energy price models and problems in pricing electricity derivatives.

2 Flow commodity markets under currency change

The methodology here is based on the assumption that there exists a market on contracts for delivery of a flow commodity at any future date $\tau \in [0, T]$. To avoid argumentation problems resulting from non–storability, we agree that the prime (storable) assets of this market are τ*-agreements* which ensure the delivery of one commodity unit on future date τ. That is, the price at time t of a τ-agreement is interpreted as the price for one commodity unit which is paid at t and supplied at τ. Now we turn our attention to futures. A flow commodity future with the delivery date τ (τ*–future* in the sequel) is introduced as a conventional future contract written on final τ–agreements price. In this sense, the τ–future prices $(E_t(\tau))_{t \in [0,\tau]}$ are settled as usual prices on futures for storable underlyings. The advantage of this viewpoint is that for a realistic model, arbitrage opportunities for trading τ–agreements and their derivatives have to be ruled out. This imposes clear requirements to be satisfied when modeling τ–futures prices.

Suppose that for each $\tau \in [0, T]$ there is a τ-future whose price evolution $(E_t(\tau))_{t \in [0,\tau]}$ is assumed to follow a positive–valued adapted stochastic process realized on a complete filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0,T]})$. Assume that at the beginning $t = 0$ prices $E^*_0(\tau)$ for all future delivery times $\tau \in [0, T]$ are observed, where the initial yield curve $E^*_0(\cdot)$ is deterministic and continuous. Let us write $\mathcal{D} := \{(t, \tau) : 0 \leq t \leq \tau \leq T\}$ for chronological time pairs.

Definition 1. *A flow–commodity market with parameter* $(n, E^*(\cdot)) \in \mathbb{N} \times C[0, T]$ *is given by* $(E_t(\tau))_{(t,\tau) \in \mathcal{D}}$ *such that*

$$(6) \qquad \mathcal{D} \to \mathbb{R},\ (t, \tau) \mapsto E_t(\tau) \text{ is continuous for all } \tau \in [0, T],$$

the initial yield curve is explained

$$(7) \qquad E_0(\tau) = E^*_0(\tau) \qquad \text{for all } \tau \in [0, T],$$

arbitrage is excluded

(8) *there exists a risk–neutral measure* Q^F *equivalent to* P *such that for each* $\tau \in [0,T]$, $(E_t(\tau))_{t\in[0,\tau]}$ *follows a* Q^F*–martingale,*

changes in the yield curve are possible in the sense that

(9) *for all* $0 < t < \tau_0 < \tau_1 <, \dots, < \tau_n \le T$ *the* $\mathcal{F}_t$*-conditional distribution of* $E_{\tau_0}(\tau_0), E_{\tau_0}(\tau_1), \dots, E_{\tau_0}(\tau_n)$ *posseses almost surely a positive Lebesgue density on* $]0,\infty[^{n+1}$

The assumption (8) is justified by the following consideration. The conventional way to rule out arbitrage opportunities for τ–agreements and their European derivatives is to postulate price dynamics such that all security prices, expressed in units of savings account, are martingales with respect to some risk-neutral measure Q^F equivalent to P. In this setting, standard arguments (see [15], p. 45) imply that futures prices have to satisfy (8).

Let us explain why the property (9) ensures that there is no *deterministic* inter–relation between prices for commodities delivered at different times $\tau_0 <, \dots, < \tau_n$ and so reflects the absolute non-storability of the commodity. Given future dates $0 < t < \tau_0 < \dots < \tau_n \le T$, let $g_t :]0,\infty[^{n+1} \to \mathbb{R}_+$ be the $\mathcal{F}_t$–conditional Lebesgue density of $(E_{\tau_0}(\tau_0), \dots, E_{\tau_0}(\tau_n))$. By assumption, $g_t(x) > 0$ for all $x \in [0,\infty[^{n+1}$ almost surely which implies that for any choice of non–empty open intervals $I_0, \dots, I_n \subset]0,\infty[$ the yield curve $(E_{\tau_0}(\tau))_{t\in[\tau_0,T]}$ at time τ_0 passes with positive probability through $(\{\tau_k\} \times I_k)_{k=0}^n$:

$$P(E_{\tau_0}(\tau_0) \in I_0, \dots, E_{\tau_0}(\tau_n) \in I_n \,|\, \mathcal{F}_t) = \int_{I_n} \dots \int_{I_0} g_t(x_0, \dots, x_n) dx_0, \dots, dx_n > 0$$

almost surely. This means that the event $\{E_{\tau_0}(\tau_0) \in I_0, \dots, E_{\tau_0}(\tau_n) \in I_n\}$ occurs with a positive probability even if we choose I_{k+1} arbitrarily far above I_k $(k = 0, \dots, n-1)$. For a storable commodity, such behavior would be impossible, since the price of a forward with a delivery date τ_{k+1} can not exceed the price of a forward with a delivery date τ_k plus costs for commodity storage during $[\tau_k, \tau_{k+1}]$. Further, it shall be pointed out that (9) excludes the *deterministic* inter–relation between futures prices. The existing correlation between futures prices with different delivery dates which result from seasonalities, long-term reduction in the power resources, and global market changes are still reflected by the non-factorizing density g_t of the $\mathcal{F}_t$–conditional distribution of $(E_{\tau_0}(\tau_0), \dots, E_{\tau_0}(\tau_n))$.

For practical implementation, n is chosen such that the model fits the real-world market. For example, the European Energy Exchange lists futures prices for electricity delivered within each of the next six calender months. That is, choosing the flow commodity unit equal to 1 MWh steadily supplied within next 30 days, it suffices to set up a model with $n = 6$.

To the best of the authors knowledge, all electricity spot price models considered in the literature do not describe independent price evolution of futures with different delivery dates in the sense of (9). Actually, the problem here is that on one hand, we would like to disconnect price dynamics $(E_t(\tau))_{t\in[0,\tau_1]}$, $(E_t(\tau))_{t\in[0,\tau_2]}$ for different $\tau_1 \neq \tau_2$ and on the other hand, we expect that $E_{t_1}(\tau_1)$ is around $E_{t_2}(\tau_2)$ if (t_1, τ_1) is close to (t_2, τ_2). Similar situation appears for zero bond prices and is successfully treated by the interest rate theory, whose methodology we transfer into the framework of flow commodity markets.

Let us comment here on a related approach followed in [11], [23], and [18], where authors address commodities with limited storability and apply convenience yield, which describes the instantaneous flow of services that accrues to the holder of the physical commodity, but not to the owner of a contract for future delivery (see [4]). In [18], a general model is presented that connects spot and futures prices via stochastic convenience yield. However, this methodology does not directly apply to commodities without any storage opportunity (like electricity) due to the lack of appropriate justification for the convenience yield. Our contribution could provide a missing link here: by establishing an explicit equivalence between flow commodity markets and money markets in the Heath-Jarrow-Morton formulation, we obtain a relation between futures and spot prices similar to that studied in [18]. Still, in our context, the absolute non-storability reflected by (9) also imposes additional requirements which have to be respected.

Given the observed yield curve $(E^*_0(\tau))_{\tau\in[0,T]}$, the crucial task is to model the entire futures price dynamics $(E_t(\tau))_{(t,\tau)\in\mathcal{D}}$ such that (6) — (9) are fulfilled. In other words, we are concerned with the problem of the *explicit construction* of flow commodity markets. It turns out that a currency change provides a solution. Namely, put the new currency unit at time t equal to one commodity unit delivered at t, then all τ–futures finish at one (exactly as zero bonds), whereas the riskless asset is transformed to a risky security. That is, such a currency change remodels our flow commodity market into a bond market equipped with an additional risky asset (let us call such a market *money market* in the sequel). On the other hand, we also learn that a money market is transformed back into a flow commodity market by the reverse currency change. As a result, we obtain a one-to-one correspondence between flow commodity markets and money markets. Utilizing the Heath-Jarrow–Morton (HJM) description of money markets, this concept finally yields explicit models of flow commodity markets. To proceed in this way, the notion of money market is introduced.

Suppose that for each $\tau \in [0,T]$ there exists is a zero bond maturing at this date whose price evolution is denoted by $(p_t(\tau))_{t\in[0,\tau]}$ and is assumed to follow a positive–valued adapted stochastic process realized on a complete filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$. We agree that the initial bond curve $p_0^*(\cdot)$ on $[0,T]$ is deterministic and continuous and suppose that there exists an additional risky asset with positive–valued adapted price process denoted by $(N_t)_{t\in[0,T]}$.

Definition 2. *A money market with parameter* $(n, p_0^*(\cdot), N_0^*) \in \mathbb{N} \times C[0,T] \times \mathbb{R}_+$ *is given by* $(p_t(\tau))_{(t,\tau)\in\mathcal{D}}, (N_t)_{t\in[0,T]}$ *such that*

(10)

$$p_t(t) = 1 \text{ for all } t \in [0,T] \text{ and } t \mapsto N_t,\ (t,\tau) \mapsto p_t(\tau) \text{ are a.s. continuous,}$$

the initial values are explained

$$N_0 = N_0^*, \quad p_0(\tau) = p_0^*(\tau) \qquad \text{for all } \tau \in [0,T], \tag{11}$$

arbitrage is excluded

(12) *there exists a positive–valued adapted discounting process* $(C_t)_{t\in[0,T]}$ *and a risk-neutral measure* Q^M *equivalent to* P *such that* $(N_t/C_t)_{t\in[0,T]}$, $(p_t(\tau)/C_t)_{t\in[0,\tau]}$ *are* Q^M*–martingales for all* $\tau \in [0,T]$,

and

(13) *for all* $0 < t < \tau_0 < \tau_1 \dots < \tau_n \le T$ *the* $\mathcal{F}_t$*-conditional distribution of* $N_{\tau_0}, p_{\tau_0}(\tau_1), \dots, p_{\tau_0}(\tau_n)$ *almost surely posseses positive Lebesgue density on* $]0,\infty[^{n+1}$

As mentioned above, there is no obvious way for canonical construction of a flow commodity market, whereas for modeling bond markets, one can rely on a well established theory of interest rate models. Considering this, we apply part (ii) of the next theorem to assemble flow commodity markets from money markets. Note that such an approach seems sufficiently general, since due to (i) of the theorem below, each commodity market is reached from an appropriate money market.

Theorem 1. *(i) Let* $(E_t(\tau))_{(t,\tau)\in\mathcal{D}}$ *be a flow commodity market with* $(n, E_0^*(\cdot))$*, then*

$$N_t := E_t(t)^{-1} \quad \text{for all } t \in [0,T], \tag{14}$$

$$p_t(\tau) := E_t(\tau) N_t \quad \text{for all } (t,\tau) \in \mathcal{D}, \tag{15}$$

gives a money market with $(n, E_0^*(\cdot)/E_0^*(0), E_0^*(0)^{-1})$*, discounting process* $(p_t(T))_{t\in[0,T]}$*, and risk-neutral measure* $dQ^M = E_T(T)E_0(T)^{-1}dQ^F$*.*
(ii) Let $(p_t(\tau))_{(t,\tau)\in\mathcal{D}}$*,* $(N_t)_{t\in[0,T]}$ *be a money market with* $(n, p_0^*(\cdot), N_0^*)$*, discounting process* $(C_t)_{t\in[0,T]}$*, and risk-neutral measure* Q^M*. Define*

$$E_t(\tau) := p_t(\tau)/N_t \quad \text{for all } (t,\tau) \in \mathcal{D}, \tag{16}$$

Then $(E_t(\tau))_{(t,\tau)\in\mathcal{D}}$ *gives a flow commodity market with* $(n, p_0^*(\cdot)/N_0^*)$ *and risk-neutral measure*

$$dQ^F = \frac{N_T}{C_T}\frac{C_0}{N_0}dQ^M \tag{17}$$

Proof. (i) The properties (10) and (11) are consequences of (6) and (7) due to (14) and (15). To show (13), we introduce the function

$$\Psi :]0,\infty[^{n+1} \to]0,\infty[^{n+1}, \qquad (y_0,\ldots,y_n) \mapsto (\frac{1}{y_0}, \frac{y_1}{y_0} \ldots, \frac{y_n}{y_0}).$$

satisfying

$$(N_{\tau_0}, p_{\tau_0}(\tau_1), \ldots, p_{\tau_0}(\tau_n)) = \Psi(E_{\tau_0}(\tau_0), \ldots, E_{\tau_0}(\tau_n)). \tag{18}$$

Since Ψ is a diffeomorphism (bijection, Ψ and Ψ^{-1} continously differentiable) and the Lebesgue density of $(E_{\tau_0}(\tau_0), E_{\tau_0}(\tau_1), \ldots, E_{\tau_0}(\tau_n))$ exists and is positive due to (9), the Lebesgue density of (18) also exists and is also positive. To prove (12), we make use of the change–of–numeraire technique (see [10], [3]), in our context it is applied as follows: For positive–valued adapted processes $(H_t)_{t\in[0,\tau]}$, $(D_t)_{t\in[0,T]}$ and $(D'_t)_{t\in[0,T]}$ holds

$$\left\{ \begin{array}{l} (H_t/D_t)_{t\in[0,\tau]} \text{ and } (D'_t/D_t)_{t\in[0,T]} \text{ are martingales wit respect to} \\ Q \text{ if and only if } (H_t/D'_t)_{t\in[0,\tau]} \text{ and } (D_t/D'_t)_{t\in[0,T]} \text{ are martingales} \\ \text{with respect to } Q' \text{ given by } dQ' = \frac{D'_T}{D_T}\frac{D_0}{D'_0}dQ. \end{array} \right. \tag{19}$$

Put now $H_t := E_t(\tau)$ for all $t \in [0,\tau]$, $D_t := 1$ $D'_t = E_t(T)$ for all $t \in [0,T]$. Then (14), (15) and (19) show that

$$(20) \qquad \left(\frac{E_t(\tau)}{E_t(T)} = \frac{p_t(\tau)}{p_t(T)}\right)_{t\in[0,\tau]} \quad \text{and} \quad \left(\frac{1}{E_t(T)} = \frac{N_t}{p_t(T)}\right)_{t\in[0,T]}$$

are martingales with respect to Q^M.

(ii) The properties (6) and (7) are consequences of (10) and (11) by definition (16). To show (9), we the same argumentation is applied as in (i) for (13), where we have to replace Ψ by Ψ^{-1}. Finally, the measure Q^F in (8) is obtained from Q^M in (12) using (17) by change of numeraire (19). □

3 Market construction with HJM approach

It will be now illustrated how the Gaussian Heath-Jarrow-Morton (HJM) interest rate models provide a starting point for the construction of flow commodity markets. Let us begin with complete filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$ where the filtration is the augmentation (by the null sets in $\mathcal{F}^W_T$) of the filtration $(\mathcal{F}^W_t)_{t\in[0,T]}$ generated by the d–dimensional Brownian motion $(W_t)_{t\in[0,T]}$. All processes are supposed to be progressively measurable. Assume that we have observed the initial curve

$$(21) \qquad (E^*_0(\tau))_{\tau\in[0,T]} \quad \text{deterministic, absolutely continuous.}$$

Specify the *forward rate volatility* $(\sigma_t(\tau))_{(t,\tau)\in\mathcal{D}}$ choosing a deterministic function

$$(22) \qquad \mathcal{D} \to \mathbb{R}^d,\ (t,\tau) \mapsto \sigma_t(\tau) \quad \text{with} \quad \int_0^T \int_0^\tau \|\sigma_t(\tau)\|^2 dt d\tau < \infty$$

and define bond volatilities by

$$(23) \qquad \overline{\sigma}_t(\tau) := \int_t^\tau \sigma_t(s) ds \quad \text{for all } (t,\tau) \in \mathcal{D}$$

Introduce the initial forward rates

$$(24) \qquad f^*_0(t) = -\frac{\partial}{\partial\tau} \ln E^*_0(\tau) \quad \text{for all } t \in [0,T]$$

to define for all $(t,\tau) \in \mathcal{D}$ the forward rates as

$$(25) \qquad f_t(\tau) = f^*_0(\tau) + \int_0^t \sigma_s(\tau) \circ \overline{\sigma}_s(\tau) ds + \int_0^t \sigma_s(\tau) dW_s,$$

and the bond price dynamics for all $\tau \in [0,T]$ as solution to

$$dp_t(\tau) = p_t(\tau)(f_t(t)dt - \overline{\sigma}_t(\tau)dW_t), \qquad p_0(\tau) = p_0^*(\tau) := E_0^*(\tau)/E_0^*(0). \tag{26}$$

Moreover, describe the evolution $(N_t)_{t\in[0,T]}$ of the additional risky asset as

$$dN_t = N_t(f_t(t)dt + v_t dW_t) \qquad N_0 := E_0^*(0)^{-1}, \tag{27}$$

with a pre–specified d–dimensional deterministic volatility

$$(v_t)_{t\in[0,T]} \quad \text{with} \quad \int_0^T \|v_s\|^2 ds < \infty.$$

Define also

$$\Sigma_t(\tau) := -\overline{\sigma}_t(\tau) - v_t \quad \text{for all } (t,\tau) \in \mathcal{D}. \tag{28}$$

$$\begin{cases} \text{Let } n \in \mathbb{N} \text{ be such that for all} \\ 0 < t < \tau_0 <, \dots, \tau_n \leq T \text{ the functions} \\ [t, \tau_0] \to \mathbb{R}^d, \quad s \mapsto \Sigma_s(\tau_i),\ (i = 0, \dots, n) \\ \text{are linearly independent.} \end{cases} \tag{29}$$

Moreover, introduce the discounting process

$$C_t := \exp(\int_0^t f_s(s)ds) \quad \text{for all } t \in [0,T]. \tag{30}$$

Using standard results from interest rate theory, we verify

Theorem 2. *For $(p_t(\tau))_{(t,\tau)\in\mathcal{D}}$, $(N_t)_{t\in[0,T]}$ from (26), (27) define $E_t(\tau) := p_t(\tau)/N_t$ for all $(t,\tau) \in \mathcal{D}$. Then $(E_t(\tau))_{(t,\tau)\in\mathcal{D}}$ gives a flow commodity market with $(n, E_0^*(\cdot))$ from (29) and (21). Moreover, the risk-neutral measure satisfies*

$$dQ^F = \exp(\int_0^T v_s dW_s - \frac{1}{2}\int_0^T \|v_s\|^2 ds)dP$$

and futures prices follow

$$dE_t(\tau) = \Sigma_t(\tau)E_t(\tau)dW_t^F, \quad E_0(\tau) = E_0^*(\tau) \tag{31}$$

with Q^F–Brownian motion

$$W_t^F := -\int_0^t v_s ds + W_t, \qquad t \in [0,T] \tag{32}$$

Proof. According to the previous theorem, it suffices to show that (26) and (27) define a money market with parameter $(n, E^*_0(\cdot)/E^*_0(0), E^*_0(0)^{-1})$, discounting process (30) and risk-neutral measure $Q^M := P$. The assumptions (10) and (11) hold due to definition (26), where to see $p_\tau(\tau) = 1$ for all $\tau \in [0, T]$, we use

$$p_t(\tau) = \exp(-\int_t^\tau f_t(s)ds) \qquad \text{for } (t, \tau) \in \mathcal{D},$$

(see Lemma 13.1.1, from [19]). Now, we prove (12) by verifying that $(N_t/C_t)_{t\in[0,]}$ and $(p_t(\tau)/C_t)_{t\in[0,\tau]}$ are martingales with respect to $Q^M := P$ as, by Ito formula, they admit stochastic differentials

$$(33) \qquad d\left(\frac{N_t}{C_t}\right) = v_t\left(\frac{N_t}{C_t}\right)dW_t, \qquad d\left(\frac{p_t(\tau)}{C_t}\right) = -\overline{\sigma}_t(\tau)\left(\frac{p_t(\tau)}{C_t}\right)dW_t.$$

Next, we prove ((9) for $(E_t(\tau))_{(t,\tau)\in\mathcal{D}}$. For any delivery date $\tau \in [0, T]$, using (26), (27) and the Ito formula, we see (31)

$$\begin{aligned} dE_t(\tau) &= d\left(\frac{p_t(\tau)}{N_t}\right) = E_t(\tau)(-v_t \circ \Sigma_t(\tau)dt + \Sigma_t(\tau)dW_t) \\ &= \Sigma_t(\tau)E_t(\tau)dW^F_t, \quad E_0(\tau) = E^*_0(\tau), \end{aligned}$$

and Girsanov theorem shows that (32) is in fact a Brownian motion under

$$dQ^F = \frac{N_T}{C_T}\frac{C_0}{N_0}dQ^M = \exp(\int_0^T v_s dW_s - \frac{1}{2}\int_0^T \|v_s\|^2 ds)dQ^M.$$

Obviously, the solution to (31) is

$$(34) \quad E_t(\tau) = E^*_0(\tau)\exp(\int_0^t \Sigma_s(\tau)dW^F_s - \frac{1}{2}\int_0^t \|\Sigma_s(\tau)\|^2 ds) \quad \text{for all } s \in [0, \tau]$$

and so the $\mathcal{F}_t$–distribution with respect to Q^F of the random variable

$$(35) \qquad (\ln E_{\tau_0}(\tau_0), \ldots, \ln E_{\tau_0}(\tau_n))$$

is Gaussian and non–degenerated due to the linear independence (29), which yields the assertion (9) since Q^F is equivalent to P. □

Example 1 Let us illustrate the above construction considering the simplest model with two-dimensional ($d = 2$) Brownian motion $(W_t = (W^1_t, W^2_t))_{t\in[0,T]}$. Specifying in (22) the constant and deterministic forward rate volatility

$$\sigma_t(\tau) := [\sigma^*, 0] \qquad \text{for all } (t, \tau) \in \mathcal{D} \text{ with given } \sigma^* \in]0, \infty[,$$

we obtain bond volatilities from (23) as

$$\overline{\sigma}_t(\tau) = [\sigma^*(\tau - t), 0] \qquad \text{for all } (t, \tau) \in \mathcal{D}. \tag{36}$$

According to (27), the additional risky asset is determined by its volatility process $(v_t)_{t \in [0,T]}$. Suppose here a constant volatility, such that the additional asset dynamics admits a correlation to bond prices:

$$v_t := [v^* \rho, v^* \sqrt{1 - \rho^2}] \qquad \text{for all } t \in [0, T]$$

with a given constant volatility parameter $v^* \in]0, \infty[$ and a correlation parameter $\rho \in [-1, 1]$. We determine n as in (29). For $0 < t < \tau_0 < \tau_1 <, \dots, < \tau_n \leq T$ we have for all $s \in [t, \tau_0]$:

$$\Sigma_s(\tau_i) \;=\; -\overline{\sigma}_s(\tau_i) - v_s = -[\sigma^*(\tau_i - s) + v^* \rho, v^* \sqrt{1 - \rho^2}] \quad \text{for } i = 0, \dots, n.$$

This shows that $n = 1$ since we can choose two $i = 0, 1$ and not more than two linearly independent functions $(s \mapsto \Sigma_s(\tau_i))_{s \in [t, \tau_0]}$. At least, the model is able to capture transitions between backwardation and contango in the flow commodity market.

4 Production in a complete forward market

Suppose that we are given a flow commodity market constructed with HJM methodology as in the previous section with $(\sigma_t(\tau))_{(t,\tau) \in \mathcal{D}}$, $(v_t)_{t \in [0,T]}$. Assume that the money short rate $(r_t)_{t \in [0,T]}$ is non-negative and introduce $(B_t = \exp(\int_0^t r_s ds))_{t \in [0,T]}$ as the savings account. For any process $(F_t)_{t \in [0,T]}$ we agree to write $(\widehat{F}_t := F_t / B_t))_{t \in [0,T]}$ for prices expressed in units of savings account. Let us consider a case where for the availability time $[0, \vartheta]$ of the virtual production capacity, the agent is able to select flow commodity futures with delivery dates $\tau_1 <, \dots, < \tau_d \in]\vartheta, T]$ to capture all price uncertainties by appropriated investments in these assets. More precisely, we focus on the case where for given $0 < \vartheta < \tau_1 <, \dots, < \tau_d \leq T$

(37)

$\Sigma_t(\tau_1), \dots, \Sigma_t(\tau_d)$ are linearly independent for Lebesgue-almost all $t \in [0, \vartheta]$.

Example 2 For two–factor model of the Example 1, the property (37) is fulfilled since for arbitrary $0 < \vartheta < \tau_1 < \tau_2 \leq T$ we see that

$$\begin{aligned}\Sigma_t(\tau_1) &= [\sigma^*(\tau_1 - t) + v^*\rho, v^*\sqrt{1-\rho^2}], \\ \Sigma_t(\tau_2) &= [\sigma^*(\tau_2 - t) + v^*\rho, v^*\sqrt{1-\rho^2}]\end{aligned}$$

are linearly independent for all $t \in [0, \vartheta]$.

For the reminder of this section, we suppose that (37) holds and introduce for almost all $t \in [0, \vartheta]$ the invertible matrix Σ_t whose rows consist of vectors

$$\Sigma_t(\tau_1), \ \ldots, \ \Sigma_t(\tau_d).$$

Then Lemma 6.7, p. 24 from [15] ensures that for each Q^M–martingale $(x + M_t)_{\in[0,\vartheta]}$ starting at $x \in \mathbb{R}$ there exists a $\mathbb{R}^d$–valued process $\pi(M) := (\pi_t = (\pi_t^1, \ldots, \pi_t^d))_{t\in[0,\vartheta]}$ such that

$$x + M_t = x + \int_0^t \pi_s \Sigma_s dW_s^F \qquad t \in [0, \vartheta]. \tag{38}$$

On the other hand, interprete $\pi(M)$ such that

$$p_s^i := \pi_s^i B_s E_s(\tau_i)^{-1}, \qquad i = 1, \ldots, d$$

are positions in futures contracts at time $s \in [0, \vartheta]$. With this interpretation, we see that starting with initial capital x, the wealth of such a strategy at time t is

$$X_t^{x,\pi(M)} = B_t(x + \sum_{i=1}^d \int_0^t p_s^i B_s^{-1} dE_s(\tau_i)) = B_t(x + M_t)$$

meaning that for any centered Q^F–martingale $M = (M_s)_{s\in[0,\vartheta]}$ there exists futures trading strategy $\pi(M)$ such that

$$\widehat{X}_t^{x,\pi(M)} = x + M_t, \quad t \in [0, \vartheta]. \tag{39}$$

Moreover, given centered martingale M, the positions $\pi_t(M)$ at any time $t \in [0, \vartheta]$ are determined from the past values $(M_s)_{s\in[0,t]}$ More precisely, we emphasize that $M \mapsto \pi(M)$ is non–anticipating. (A mapping J acting on adapted processes is non–anticipating, if for all $t \in [0, \vartheta]$ and for arbitrary h, h' in the domain of J from $\mathbb{I}_{[0,t]}h = \mathbb{I}_{[0,t]}h'$ it follows that $\mathbb{I}_{[0,t]}Jh = \mathbb{I}_{[0,t]}Jh'$, here $\mathbb{I}_{[0,t]}$ stands for a stochastic process whose paths are indicator functions of the interval $[0, t] \subseteq [0, T]$).

Now we turn to valuation of production capacity. Let us point out that in the framework of HJM-modeling, the total discounted return of virtual production capacity

$$\overline{R} := \int_0^{\vartheta} \widehat{R}_s ds \text{ is integrable with respect to } Q^F, \tag{40}$$

since, according to (21), we have

$$\int_0^{\vartheta} E_{Q^F}(\widehat{R}_s)ds = \int_0^{\vartheta} E_{Q^F}((E_s(s)-K)^+/B_s)ds \leq \int_0^{\vartheta} E_{Q^F}(E_s(s))ds = \int_0^{\vartheta} E_0^*(s)ds < \infty.$$

For given $\lambda, \Lambda \in]0, \infty[$, we choose the set of progressively measurable processes

$$\mathcal{U} = \{q = (q_t)_{t\in[0,\vartheta]} \ : \ [0, \lambda]\text{–valued, with } \int_0^{\vartheta} q_s ds \leq \Lambda \ \}$$

to represent all admissible exercise policies.

Proposition 1. *Under the assumptions of this section holds: if the initial price of production capacity is different from*

$$x_0 = \sup\{E_{Q^F}(\int_0^{\vartheta} q_s \widehat{R}_s ds) \ : \ q \in \mathcal{U}\}, \tag{41}$$

then there exists an arbitrage opportunity.

Proof. Suppose that the production capacity is offered at a price $x_0' < x_0$. Then there is a *long arbitrage*: the agent enters a long position at x_0' and exercises the contract by a policy $q^* \in \mathcal{U}$ with

$$x_0' < x_0^* := E_{Q^F}(\int_0^{\vartheta} q_t^* \widehat{R}_t dt) \leq x_0$$

which ensures a cash–flow at intensity $(q_t^* R_t)_{t\in[0,\vartheta]}$. Simultaneously, the agent writes a contingent claim promising the same cash–flow. Due to replication property (39), the market will pay x_0^* for this claim. Thus, the agent takes the arbitrage $x_0^* - x_0' > 0$.

If the contract is asked at a price $x_0' > x_0$, then there is a *short arbitrage*. The crucial point is to show that

Lemma 1. *There exists a correspondence $u \mapsto M^u$ mapping $\mathcal{U}$ into the set of centered martingales on $[0, \vartheta]$ which is non–anticipating and fulfills*

$$\int_0^t u_s \widehat{R}_s ds \leq x_0 + M_t^u \quad \textit{for all } t \in [0, \vartheta] \textit{ for each } u \in \mathcal{U}. \tag{42}$$

Using this result (for proof, see Appendix), we find a short arbitrage as follows: The agent enters a short position to receive x_0', then the part x_0 with $0 < x_0 < x_0'$ of this capital is used to start futures trading strategy $\pi(M^u)$ whose discounted wealth equals to the right–hand side of (42). Note that the wealth of this strategy covers all agents liabilities by the inequality in (42) and that $\pi(M^u)$ is tractable in the sense that its asset allocation at any time t depends on the long party's exercise policy $(u_s)_{s\in[0,t]}$ until t, since $u \mapsto M^u \mapsto \pi(M^u)$ is non–anticipating. As a result, the agent takes arbitrage $x_0' - x_0 > 0$. □

5 Valuation of a hydro electric power plant

To illustrate the use of our approach, we focus on the virtual production capacity and show that for the case of zero production cost, the volatility of the additional asset does not enter the price of the virtual production capacity. This is a useful simplification, meaning that for the case of hydro electric power plant (where $K = 0$), we merely need to estimate the forward rate volatility.

Suppose we are given a flow commodity market constructed from Gaussian HJM-model such that (37) holds.

Proposition 2. *The initial price (41) for the virtual production capacity with strike price* $K = 0$ *is given by*

$$x_0 = E_0^*(0) \sup_{q\in\mathcal{U}} \int_0^{\vartheta} E_{Q^M}(q_t/C_t)dt \tag{43}$$

Proof. Due to Fubini theorem, it suffices to verify for each $t \in [0, \vartheta]$ that

$$E_{Q^F}(q_t E_t(t)) = E_0^*(0) E_{Q^M}(q_t/C_t) \quad \text{for any } \mathcal{F}_t\text{–measurable bounded } q_t.$$

This equality is derived using HJM–construction as

$$\begin{aligned} E_{Q^F}(q_t E_t(t)) &= E_{Q^F}(\frac{q_t}{N_t}) = E_{Q^M}(\frac{q_t}{N_t}\frac{N_T}{C_T}\frac{C_0}{N_0}) \\ &= \frac{C_0}{N_0} E_{Q^M}(\frac{q_t}{N_t}\frac{N_t}{C_t}) = E_0^*(0) E_{Q^M}(\frac{q_t}{C_t}) \end{aligned} \tag{44}$$

□

As an illustration of the above result, we discuss the value of the virtual hydro power plant for the two-factor model of the Example 1. Here, the forward rate equals to $f_s(s) = f_0(s) + \frac{1}{2}\sigma^{*2}s^2 + \sigma^* W_s^1$ for all $s \in [0,T]$. Hence

$$\begin{aligned} C_t^{-1} &= \exp(-\int_0^t f_s(s)ds) = p_0^*(t)\exp(-\int_0^t \frac{1}{2}\sigma^{*2}s^2 + \sigma^* W_s^1 ds) \\ &= \frac{E_0^*(t)}{E_0^*(0)}\exp(-\sigma^* t W_t)\exp(\int_0^t \sigma^* s dW_s^1 - \frac{1}{2}\int_0^t (\sigma^* s)^2 ds) \end{aligned}$$

Using the Girsanov transform, introduce the new measure $\breve{Q}$ and a $\breve{Q}$–Brownian motion by

$$\begin{aligned} d\breve{Q} &= \exp(\int_0^T \sigma^* s dW_s^1 - \frac{1}{2}\int_0^T (\sigma^* s)^2 ds) dQ^M, \\ \breve{W}_t &= W_t^1 - \int_0^t \sigma^* s ds \qquad \text{for all } t \in [0,T], \end{aligned}$$

to rewrite (44) as

$$E_0^*(0)E_{Q^M}(\frac{q_t}{C_t}) = E_{\breve{Q}}(q_t E_0^*(t)\exp(-\frac{\sigma^{*2}}{2}t^3 - \sigma^* t\breve{W}_t)).$$

That is, the initial price of hydro storage is

$$x_0 = \sup_{q \in \mathcal{U}} E_{\breve{Q}}(\int_0^\vartheta q_t H(t, \breve{W}_t)dt) \tag{45}$$

with function H given by

$$H(t,w) = E_0^*(t)\exp(-\frac{\sigma^{*2}}{2}t^3 - \sigma^* t w) \qquad \text{for all } t \in [0,\vartheta],\ \ w \in \mathbb{R}.$$

Suppose now that there exists a sufficiently smooth $V : [0,T] \times [0,\Lambda] \times \mathbb{R} \to \mathbb{R}$ representing the so–called value function as

$$V(t,\rho,\breve{W}_t) = \sup\{E_{\breve{Q}}(\int_t^\vartheta q_s H(s,\breve{W}_s)ds \,|\mathcal{F}_t) \,:\, q \in \mathcal{U},\ \int_t^\vartheta q_s ds \le \rho\}. \tag{46}$$

With this assumption, the price for the virtual power plant is $x_0 = V(0,\Lambda,0)$. The corresponding Hamilton–Bellman–Jacobi equation on $]0,T[\times]0,\Lambda[\times\mathbb{R}$ is

$$\max_{q \in \{0,\lambda\}} q\left(H(t,w) - \frac{\partial V}{\partial \rho}(t,\rho,w)\right) + \frac{\partial V}{\partial t}(t,\rho,w) + \frac{1}{2}\frac{\partial^2}{\partial w^2}V(t,\rho,w) = 0 \tag{47}$$

subject to boundary conditions

$$V(T,\rho,w)=0 \quad \text{for all } (\rho,w)\in[0,\Lambda]\times\mathbb{R},$$
$$V(t,0,w)=0 \quad \text{for all } (t,w)\in[0,T]\times\mathbb{R}.$$

Since no closed–form solution to (47) is available, let us elaborate on the corresponding trinomial tree model.

Consider within $[0,\vartheta]$ discrete equidistant times $(k\frac{\vartheta}{N})_{k=0}^{N-1}$ and approximate in distribution the Brownian motion $(\breve{W}_t)_{t\in[0,\vartheta]}$ on these times by a random walk

$$\breve{W}_{k\frac{\vartheta}{N}} \approx \sqrt{\frac{\vartheta}{N}\frac{3}{2}}\breve{w}(k) \text{ for } k=0,\dots,N-1 \tag{48}$$

such that $\breve{w}(0)=0$ with independent increments $(\breve{w}(j)-\breve{w}(j-1))_{j=0}^{N-1}$ uniformly distributed on $\{-1,0,1\}$ each. Let us restrict ourselves to consider admissible policies $(q(k))_{k=0}^{N-1}$ taking a finite number $(L+1)\in\mathbb{N}$ of values $\{0,\lambda/L,\dots,L\lambda/L\}\subset[0,\lambda]$ and adapted to the filtration generated by the left-hand side of (48). That is, in the discrete–time model we have to replace $\mathcal{U}$ by

(49)

$$\mathcal{U}_N := \{(q(k))_{k=0}^{N-1} \ : \ (\mathcal{G}_k)_{k=0}^{N-1}\text{–adapted, } \{0,\dots,L\}\text{-valued, } \textstyle\sum_{k=0}^{N-1} q(k)=\rho_{max}\},$$

where $(\mathcal{G}_k)_{k=0}^{N-1}$ is the filtration generated by the right-hand side of (48) and ρ_{max} is the integer part of $\Lambda LN/(\lambda\vartheta)$, which comes from the constraint (3), in discrete-time model transformed to

$$\sum_{k=0}^{N-1} q(k)\frac{\lambda}{L}\frac{\vartheta}{N}\le\Lambda.$$

With these conventions, we approximate the integral in (45) by

$$x_0^N := \sup\{E(\sum_{k=0}^{N-1} q(k)H^N(k,\breve{w}(k))) \ : \ (q(k))_{k=0}^{N-1}\in\mathcal{U}_N\} \tag{50}$$

where H^N is given by

$$H^N(k,w) := \frac{\lambda}{L}\frac{\vartheta}{N}H(\frac{\vartheta}{N}k,\sqrt{\frac{\vartheta}{N}}w) \qquad k=0,\dots N-1, \quad w=-k,\dots,k.$$

Obviously, (50) defines a standard discrete–time optimal control problem. The corresponding value function

(51)

$$V^N(k,\rho,w)=\sup\{E(\textstyle\sum_{j=k}^{N-1} q(j)H^N(j,\breve{w}(j))\,|\,\breve{w}(k)=w) \ : \ \sum_{j=k}^{N-1} q(j)=\rho\}$$
$$\text{for } k=0,\dots,N-1, \ \ w=-k,\dots,k, \ \ \rho=0,\dots,\rho_{max}$$

solves (50) by $V^N(0,\rho_{max},0)$. The values (51) are obtained from the following recursion for $k=1,\dots,N$

$$(52)\quad V^N(k-1,\rho,w)=\max_{l=0,\dots,L\wedge\rho}\left(lH^N(k-1,w)+\frac{1}{3}\sum_{j=-1}^{1}V^N(k,\rho-l,w+j)\right)$$

started at

$$V^N(N,w,\rho)=0 \qquad \text{for all } w=-N,\dots,N,\ \rho=0,\dots,\rho_{max}.$$

As an application, we shall discuss the dependence of energy price on the electrical power of production capacity.

Example 3 In practice, electricity retailers are obliged to cover the random demand of their final customers at a fixed price. To fulfill such agreements, they purchase appropriate physical and/or virtual production capacities. A common problem hereby is to estimate the sensitivity of the plants value with respect to electrical power λ. The qualitative behavior is obvious: For fixed energy amount Λ and availability period ϑ, the higher the electrical power λ, the more flexible and thus more valuable the contract. However, only a reliable quantitative estimate can finally answer the question, if by upgrading the plant (say, through a costly installation of additional turbines), the producer can increase λ such that all investments are fully rewarded by the market. For example, a large Swiss power producer was restricted to run a turbine at 25 MW, which is 5 MW below its nominal power of 30 MW, in order to respect noise protection for a residential house nearby. In this case, what needs to be determined is the maximum purchase price for the house, which would be still acceptable for producer.

We use the trinomial tree model from above for an exemplary discussion. Suppose we are given a virtual hydro electric power plant for $\vartheta=1$ year (365×24 hours) with the total amount of energy $\Lambda=365\times24$ MWh. Assume that the yield curve is constant at $E_0^*(\tau)=30$ EURO/MWh for the entire year $\tau\in[0,\vartheta]$ in advance and suppose, for simplicity, that the interest rate is zero. If the maximal electrical power equals to 1 MW (365×24 MWh/year), then the total value of the virtual power plant is $x_0=E_Q(\int_0^\vartheta\lambda\widehat{E}_t(t)dt)=\lambda\int_0^\vartheta E_0^*(\tau)d\tau=365\times24\times30$ EURO, with electricity price $x_0/\Lambda=30$ EURO/MWh within this contract. Now we gradually increase the electrical power λ from 1 MWh to 40 MWh and calculate the electricity price within the corresponding contracts. The Figure 1

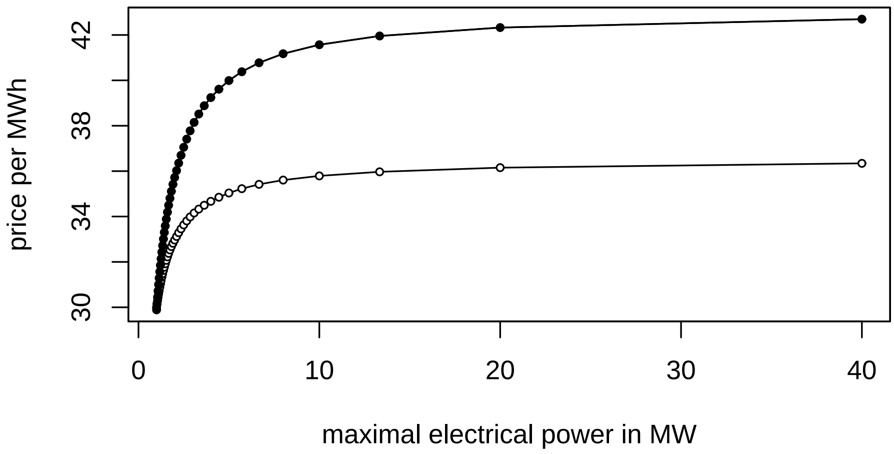

Figure 1: The dependence of energy price $x_0^N(\lambda)/\Lambda$ for $N = 40$ and $\Lambda = 1$ on the maximal electrical power λ, calculated for the forward rate volatilities $\sigma^* = 2$ (upper curve) and $\sigma^* = 1$ (lower curve).

illustrates the dependence $\lambda \mapsto x_0(\lambda)/\Lambda$ for two choices of forward rate volatilities $\sigma^* = 1, 2$, (A statistical estimation of parameter σ^* presented in [12] yields forward rates in this order of magnitude.) The dots in the graph mark calculated values, which we interpolate by straight lines. The calculation is based on (52) with $N = 40$.

6 Appendix

Let us compose standard results from the optimal control theory to verify Lemma 1. Remember that $\mathbb{I}_{[0,t]}$ denotes a stochastic process whose paths are indicator functions of the interval $[0, t]$ and then let us introduce the value function of the policy $u \in \mathcal{U}$ at time $t \in [0, \vartheta]$ as

$$V_t^u := \sup\{E_{Q^F}(\int_t^\vartheta w_s \widehat{R}_s ds \,|\, \mathcal{F}_t) \;:\; w \in \mathcal{U}, \text{ such that } \mathbb{I}_{[0,t]} w = \mathbb{I}_{[0,t]} u \;\}. \tag{53}$$

and consider

$$Z_t^u := \int_0^t u_s \widehat{R}_s ds + V_t^u, \qquad t \in [0, \vartheta]. \tag{54}$$

The main step is see that for each $u \in \mathcal{U}$

(55) $(Z_t^u)_{t\in[0,\vartheta]}$ is a supermartingale of class D starting at x_0, such that $t \mapsto E_{Q^F}(Z_t^u)$ is right–continuous

which admits (see Theorem 3.13 and Theorem 4.10 in [14]) the Doob–Meyer decomposition

$$Z_t^u = x_0 + M_t^u - A_t^u \qquad t \in [0, \vartheta] \tag{56}$$

where $(M_t^u)_{t\in[0,\vartheta]}$ is a uniformly integrable martingale and $(A_t^u)_{t\in[0,\vartheta]}$ is natural non–decreasing process, both satisfying $M_0^u = A_0^u = 0$. The decomposition (56) yields immediately the assertion of Lemma 1 due to $V_t^u \geq 0$, $A_t^u \geq 0$ for all $t \in [0, \vartheta]$. Thus, we have to prove

Lemma 2. *For each $u \in \mathcal{U}$ holds:*
(i) $(Z_t^u)_{t\in[0,\vartheta]}$ is a supermartingale,
(ii) $\{Z_\tau^u : \tau$ is a stopping time $0 \leq \tau \leq T\}$ is uniformly integrable,
(iii) $t \mapsto E_{Q^F}(Z_t^u)$ is right–continuous on $[0, \vartheta[$.

First, we prepare some notations and facts needed in the proof. Given $u \in \mathcal{U}$, we write

$$\mathcal{U}_t^u \;:=\; \{w \in \mathcal{U} \,:\, \mathbb{I}_{[0,t]} w = \mathbb{I}_{[0,t]} u\} \quad \text{for all } t \in [0, \vartheta]$$

to denote the set of admissible continuations of the exercise policy $u \in \mathcal{U}$ after t. Introduce the revenue of production capacity, expected after t for following policy $u \in \mathcal{U}$:

$$Y_t^u \;:=\; E_{Q^F}\Big(\int_t^\vartheta u_s \widehat{R}_s ds \,|\, \mathcal{F}_t\Big).$$

Let us utilize the concept of the essential supremum $\bigvee \mathcal{W}$ for a family $\mathcal{W}$ of random variables (see [15], Appendix A), to ensure that the value function in (53) is well–defined by

$$V_t^u := \bigvee \{Y_t^q \,:\, q \in \mathcal{U}_t^u\}. \tag{57}$$

Moreover, we will need the fact that for each $t \in [0, \vartheta]$ and $u \in \mathcal{U}$ the set

$$\{Y_t^q \,:\, q \in \mathcal{U}_t^u\} \tag{58}$$

is closed under pairwise maximization

$$\text{for } Y_t^q, Y_t^w \text{ with } q, w \in \mathcal{U}_t^u \text{ there exists } h \in \mathcal{U}_t^u \text{ with } Y_t^h = Y_t^q \vee Y_t^w. \tag{59}$$

Indeed, given $q, w \in \mathcal{U}_t^u$ the composed policy

$$h_s = \mathbb{I}_{[0,t]}(s)u_s + 1_{\{Y_t^q > Y_t^w\}}\mathbb{I}_{]t,\vartheta]}(s)q_s + 1_{\{Y_t^q \le Y_t^w\}}\mathbb{I}_{]t,\vartheta]}(s)w_s \qquad s \in [0, \vartheta]$$

obviously meets (59). According to [15], Lemma A.2, the Property (59) ensures that (57) is the limit of an almost surely converging non-decreasing sequence in (58) which finally implies that for any σ–algebra $\mathcal{G} \subset \mathcal{F}$ we have

$$E_{Q^F}(V_t^u \,|\, \mathcal{G}) = \bigvee\{E_{Q^F}(Y_t^q \,|\, \mathcal{G}) \,:\, q \in \mathcal{U}_t^u\} \qquad \text{for all } t \in [0, \vartheta] \text{ and } u \in \mathcal{U}. \tag{60}$$

Now, the optimality principle

$$V_{t_0}^u \ge E_{Q^F}(\int_{t_0}^t u_s \widehat{R}_s ds \,|\, \mathcal{F}_{t_0}) + E_{Q^F}(V_t^u | \,\mathcal{F}_{t_0}) \qquad \text{for all } 0 \le t_0 \le t \le \vartheta,\ u \in \mathcal{U} \tag{61}$$

is deduced from

$$\begin{aligned}
V_{t_0}^u &= \bigvee\{E_{Q^F}(\int_{t_0}^{\vartheta} w_s \widehat{R}_s ds \,|\, \mathcal{F}_{t_0}) \,:\, w \in \mathcal{U}_{t_0}^u\}, \\
&\ge \bigvee\{E_{Q^F}(\int_{t_0}^{\vartheta} w_s \widehat{R}_s ds \,|\, \mathcal{F}_{t_0}) \,:\, w \in \mathcal{U}_{t}^u\}, \qquad \text{since } \mathcal{U}_{t_0}^u \supseteq \mathcal{U}_t^u \\
&\ge \bigvee_{w \in \mathcal{U}_t^u} \Big(E_{Q^F}(\int_{t_0}^{t} w_s \widehat{R}_s ds \,|\, \mathcal{F}_{t_0}) + E_{Q^F}(\int_{t}^{\vartheta} w_s \widehat{R}_s ds \,|\, \mathcal{F}_{t_0}) \Big) \\
&\ge E_{Q^F}(\int_{t_0}^{t} u_s \widehat{R}_s ds \,|\, \mathcal{F}_{t_0}) + \bigvee_{w \in \mathcal{U}_t^u} E_{Q^F}(Y_t^w \,|\, \mathcal{F}_{t_0}) \\
&\ge E_{Q^F}(\int_{t_0}^{t} u_s \widehat{R}_s ds \,|\, \mathcal{F}_{t_0}) + E_{Q^F}(V_t^u | \,\mathcal{F}_{t_0})
\end{aligned}$$

where for the last step we have used (60). Now we prove Lemma 2.

Proof. (i) is an immediate consequence of the optimality principle (61).

(ii) By definition (54) and due to (40), we obtain a dominating martingale $(Z_t)_{t\in[0,\vartheta]}$ as

$$Z_t := \lambda E_{Q^F}(\overline{R}\,|\,\mathcal{F}_t) \geq Z_t^u \geq 0, \qquad t \in [0,\vartheta],$$

which gives for all $a > 0$ and each stopping time τ the estimate

$$E_{Q^F}(1_{\{|Z_\tau|>a\}}|Z_\tau|) \geq E_{Q^F}(1_{\{|Z_\tau^u|>a\}}|Z_\tau^u|). \tag{62}$$

On the other hand, Lemma VI.29.6 from [22] ensures the uniform integrability of

$$\{Z_\tau = E_{Q^F}(Z_\vartheta|\mathcal{F}_\tau) \,:\, \tau \text{ stopping time}, \;\; 0 \leq \tau \leq T\},$$

hence (62) implies (ii).
(iii) The argumentation is based on two inequalities, where the first is

$$|E_{Q^F}(V_t^u) - E_{Q^F}(V_t^w)| \leq \lambda E_{Q^F}(D(\rho_t^u, \rho_t^w)\overline{R}) \;\text{ for all }\; u, w \in \mathcal{U},\; t \in [0,\vartheta] \tag{63}$$

with notation

$$\rho_t^q := \Lambda - \int_0^t q_s ds \quad \text{for } q \in \mathcal{U},\; t \in [0,\vartheta]$$

for the capacity level, remaining after following policy $q \in \mathcal{U}$ until t. The difference $D(\rho_t^u, \rho_t^w)$ in capacity levels is measured by

$$D(\varrho, \varrho') := \begin{cases} 1 - \frac{\varrho}{\varrho'} & \text{if } \varrho < \varrho' \\ 1 - \frac{\varrho'}{\varrho} & \text{if } \varrho > \varrho' \\ 0 & \text{if } \varrho = \varrho' \end{cases} \qquad \text{for all }\; \varrho, \varrho' \in [0, \Lambda]$$

with the property that

$$\begin{array}{l} \text{for non-decreasing sequences } (\varrho(i))_{i\geq 1}, (\varrho'(i))_{i\geq 1} \subset [0,\Lambda] \\ \text{holds } \lim_{i\to\infty} D(\varrho(i), \varrho'(i)) = D(\lim_{i\to\infty} \varrho(i), \lim_{i\to\infty} \varrho'(i)). \end{array} \tag{64}$$

The natural interpretation of (63) is that in expectation, the revenue difference of best possible continuations of u and w depends only on their residual capacity levels at t. The other estimation we require claims that for each $u \in \mathcal{U}$, $t_0 \in [0, \vartheta[$ and $\varepsilon > 0$ there exist $w \in \mathcal{U}_{t_0}^u$ such that

$$|E_{Q^F}(V_{t_0}^u) - E_{Q^F}(V_t^w)| \leq \lambda(t - t_0)E_{Q^F}(\overline{R}) + \varepsilon \quad \text{for all } t \in [t_0, \vartheta]. \tag{65}$$

Both inequalities are used to prove the right–continuity of $t \mapsto E_{Q^F}(V_t^u)$ as follows: Combining (65) with (63), we conclude that for a given $\varepsilon > 0$ there exists $w \in \mathcal{U}_{t_0}^u$ such that

$$|E_{Q^F}(V_{t_0}^u) - E_{Q^F}(V_t^u)| \;\;\leq\;\; \lambda(t - t_0)E_{Q^F}(\overline{R}) + \varepsilon + \lambda E_{Q^F}(D(\rho_t^u, \rho_t^w)\overline{R}). \tag{66}$$

For each non-increasing sequence $(t_i)_{i\geq 1} \subset [t_0, \vartheta]$ converging to t_0 we have the non–decreasing sequences $(\rho^u_{t_i})_{i\geq 1}$ and $(\rho^w_{t_i})_{i\geq 1}$ in $[0, \Lambda]$ with

$$\lim_{i\to\infty} \rho^u_{t_i} = \rho^u_{t_0}, \ \lim_{i\to\infty} \rho^w_{t_i} = \rho^w_{t_0} \quad \text{where } \rho^u_{t_0} = \rho^w_{t_0} \text{ since } w \in \mathcal{U}^u_{t_0}.$$

By dominated convergence and (64), we deduce that the limiting behavior of (66) is

$$\limsup_{t_i \downarrow t_0} |E_{Q^F}(V^u_{t_0}) - E_{Q^F}(V^u_{t_i})| \leq \varepsilon \quad \text{for all } \varepsilon > 0.$$

This gives the assertion of (iii), since the other summand $t \mapsto \int_0^t u_s R_s ds$ in (54) is obviously continuous.

Now we prove (65). The idea is to approximate the policy $\tilde{w} \in \mathcal{U}^w_t$ on $[t, \vartheta]$ by appropriate policy $\tilde{u} \in \mathcal{U}^u_t$ constructed as

$$\tilde{u}_s := u_s \mathbb{I}_{[0,t]}(s) + \tilde{w}_s 1_{\{\rho^w_t \leq \rho^u_t\}} \mathbb{I}_{]t,\vartheta]}(s) + \frac{\rho^u_t}{\rho^w_t} \tilde{w}_s 1_{\{\rho^w_t > \rho^u_t\}} \mathbb{I}_{]t,\vartheta]}(s), \quad s \in [0, \vartheta]$$

with the interpretation to trace a foreign policy $\tilde{w} \in \mathcal{U}^w_t$ by the own policy $\tilde{u} \in \mathcal{U}^u_t$ using the following strategy: If the own capacity level ρ^u_t is greater or equal to the foreign level ρ^w_t, then we just mimic $\tilde{w}$. Otherwise, we have to follow $\tilde{w}$ at a reduced intensity ρ^u_t/ρ^w_t. As a result, we obtain the estimation

$$\begin{aligned} Y^{\tilde{w}}_t - Y^{\tilde{u}}_t &= \int_t^\vartheta \tilde{w}_s \widehat{R}_s ds - \int_t^\vartheta \tilde{u}_s \widehat{R}_s ds = (1 - \frac{\rho^u_t}{\rho^w_t}) 1_{\{\rho^w_t > \rho^u_t\}} \int_t^\vartheta \tilde{w}_s \widehat{R}_s ds \\ &\leq \lambda (1 - \frac{\rho^u_t}{\rho^w_t}) 1_{\{\rho^w_t > \rho^u_t\}} \overline{R}. \end{aligned}$$

Taking the expectation, we deduce

$$E_{Q^F}(Y^{\tilde{w}}_t) - E_{Q^F}(Y^{\tilde{u}}_t) \leq \lambda E_{Q^F}((1 - \frac{\rho^u_t}{\rho^w_t}) 1_{\{\rho^w_t > \rho^u_t\}} \overline{R}) \tag{67}$$

and passing through supremum, we conclude that

$$E_{Q^F}(V^w_t) - E_{Q^F}(V^u_t) \leq \lambda E_{Q^F}((1 - \frac{\rho^u_t}{\rho^w_t}) 1_{\{\rho^w_t > \rho^u_t\}} \overline{R}).$$

Interchange now u and w in the above argumentation, to obtain the reverse estimate

$$E_{Q^F}(V^u_t) - E_{Q^F}(V^w_t) \leq \lambda E_{Q^F}((1 - \frac{\rho^w_t}{\rho^u_t}) 1_{\{\rho^u_t > \rho^w_t\}} \overline{R}). \tag{68}$$

Combining (68) and (67) finally yields (63) by

$$
\begin{aligned}
|E_{Q^F}(V_t^w) - E_{Q^F}(V_t^u)| &\leq \lambda E_{Q^F}((1-\frac{\rho_t^u}{\rho_t^w})1_{\{\rho_t^w>\rho_t^u\}}\overline{R}) + \lambda E_{Q^F}((1-\frac{\rho_t^w}{\rho_t^u})1_{\{\rho_t^u>\rho_t^w\}}\overline{R}) \\
&\leq \lambda E_{Q^F}(((1-\frac{\rho_t^u}{\rho_t^w})1_{\{\rho_t^w>\rho_t^u\}} + (1-\frac{\rho_t^w}{\rho_t^u})1_{\{\rho_t^u>\rho_t^w\}})\overline{R}) \\
&\leq \lambda E_{Q^F}(D(\rho_t^w,\rho_t^u)\overline{R}).
\end{aligned}
$$

Now we show (65). According to (60), we can choose for a given $\varepsilon > 0$ a policy $\tilde{u} \in \mathcal{U}_{t_0}^u$ such that

$$
\begin{aligned}
E_{Q^F}(V_{t_0}^u) - \varepsilon &\leq E_{Q^F}(Y_{t_0}^{\tilde{u}}) = E_{Q^F}(\int_{t_0}^t \tilde{u}_s\widehat{R}_s ds) + E_{Q^F}(Y_t^{\tilde{u}}) \\
&\leq E_{Q^F}(\int_{t_0}^t \tilde{u}_s\widehat{R}_s ds) + E_{Q^F}(V_t^{\tilde{u}}) \qquad (69)
\end{aligned}
$$

Since $0 \leq E_{Q^F}(V_{t_0}^u) - E_{Q^F}(V_t^{\tilde{u}})$ holds for $\tilde{u} \in \mathcal{U}_{t_0}^u$ by the optimality principle (61), we rewrite (69) as (65):

$$
|E_{Q^F}(V_{t_0}^u) - E_{Q^F}(V_t^{\tilde{u}})| \leq E_{Q^F}(\int_{t_0}^t \tilde{u}_s\widehat{R}_s ds) + \varepsilon \leq \lambda(t-t_0)E_{Q^F}(\overline{R}) + \varepsilon.
$$

□

References

[1] Barlow, M. T., *A diffusion model for electricity prices.* Math. Finance, 12, 287-298, (2002).

[2] Bjerksund, P.; Rasmussen, H.; G. Stensland, *Valuation and risk management in the Nordic electricity market.* Working paper, Institute of Finance and Management Science, Norwegian School of Economics and Business Administration, (2000).

[3] Björk, T., *Interest Rate Theory.* Lecture Notes in Mathematics Vol. 1656, Springer, (1996).

[4] Brennan, M. J., *The supply of storage.* The American Economic Review, Vol 41, 50-72, (1958).

[5] Carmona, R., Durrleman V., *Pricing and hedging spread options.* SIAM Review, Vol. 45, pp. 627-685, (2003).

[6] Fama, E., F., French, K., R, *Commodity future prices: Some evidence on forecast power, premiums, and the theory of storage.* The Journal of Business, Vol. 60, Issue 1, 55-73, (1987).

[7] Burger, M., Klar, B., Müller, A. and Schindlmayr, G. . *A spot market model for the pricing of derivatives in electricity markets.* Quantitative Finance, 4, 109-122, (2003).

[8] Eydeland, A.; Geman H., *Pricing Power Derivatives*, Risk, Vol. 11, No. 10, 71–73, (1998).

[9] Eydeland, A.; Geman H. *Fundamentals of Electricity Derivatives.* Energy modelling and the management of uncertainty, Risk publications, chap. 3, 35-43, (2003).

[10] Geman H., El Karoui, N., Rochet J. C., *Changes of numeraire, changes of probability measure and option pricing.* Journal of Applied Probability, 32, 443-458, (1995).

[11] Gibson, R., Schwartz, E., S., *Stochastic convenience yield and pricing of oil contingent claims.* The Journal of Finance, Vol. 45, Issue 3, 959-976.

[12] Hinz, J., von Grafenstein, L., Verschuere, M., Wilhelm, M., *Pricing electricity risk by interest rate methods.* Quantitative Finance (accepted for publication).

[13] Hinz, J., *Modelling day–ahead electricity prices,* Applied Mathematical Finance, Vol 10, No 2, 149–161, (2003).

[14] Karatzas, I.; Shreve, S. E. *Brownian Motion and Stochastic Calculus.* Second Edition, Springer 1997.

[15] Karatzas, I. ; Shreve, S. E. , *Methods of Mathematical Finance.* Springer, New York, 1998.

[16] Koekebakker, S; Ollmar, F., *Forward curve dynamics in the Nordic electricity market.* Preprint, (2001).

[17] Schwartz, E. S.; Lucia, J. J., *Electricity prices and power derivatives. Evidence from the Nordic Power Exchange.* Review of Derivatives Research, 5(1), 5-50, (2002).

[18] Miltersen, K., R., Schwartz, E., S., *Pricing of options on commodity futures with stochastic term structures of convenience yields and interest rates.* The Journal of Financial and Quantitative Analysis, Vol. 33, Issue 1, (1998).

[19] Musiela, M.; Rutkowski, M., *Martingale Methods in Financial Modelling.* Springer, (1997).

[20] Philipovic, D., *Energy risk: Valuing and managing energy derivatives*, McGraw-Hill, New York, (1998)

[21] Schwartz, E. S., The stochastic behavior of commodity prices: Implication for valuation and hedging. Journal of Finance, Vol. 52, 922-973, (1997).

[22] L. C. G. Rogers and D. Williams. *Diffusions, Markov Processes and Martingales.* Volume 2 Cambridge University Press, second edition, 1994.

[23] Schwartz, E. S.; Lucia, J. J., *Electricity prices and power derivatives. Evidence from the Nordic Power Exchange.* Review of Derivatives Research, 5(1), 5-50, (2002).