Pricing flow commodity derivatives using fixed income market techniques

Juri Hinz
Institute for Operations Research and RiskLab
ETH Zentrum
CH-8092 Zurich, Switzerland
e-mail hinz@ifor.math.ethz.ch

Martina Wilhelm
Institute for Operations Research
ETH Zentrum
CH-8092 Zurich, Switzerland
e-mail wilhelm@ifor.math.ethz.ch

Abstract

In this work, the valuation of energy-related financial contracts written on prices of flow commodities (such as natural gas, oil and electrical power) will be elaborated. Due to restrictions on storability of the underlying, the pricing of flow commodity derivatives is not trivial and thus correct valuation is still under discussion. In this paper, an axiomatic setting is followed, which provides a connection to interest rate theory, which is utilized to consistently price frequently quoted flow commodity options such as caps, floors, collars and cross commodity spreads.

Key words: commodity options, electricity risk, energy economics, futures markets, power derivatives.

1 Introduction

Energy markets have been, or are in the process of being deregulated in many parts of the world. This fundamentally effects changes in the energy industry, where participants have to cope with uncertainties in fuel and electrical power prices. For these reasons, commodity exchanges offer short- and long-term contracts on future energy delivery, among which, futures and their derivatives enjoy a remarkable popularity. Nonetheless, accurate pricing of these instruments is under discussion. The most significant issue is the *storability restriction*, which implies that traditional valuation methods are not adequate and new approaches are required to value even the simplest energy derivatives. That is, modeling arbitrage-free flow commodity price dynamics turns out to be the crucial step towards fair pricing of financial products in energy markets.

In this text an approach is utilized which converts a flow commodity market, such as that of natural gas, oil or electricity, into a market consisting of zero bonds equipped with an additional risky asset. Using this structure, we apply the interest rate theory and change-of-numeraire techniques to derive a methodology for pricing flow commodity derivatives. As a result, quasi-explicit formulae for spread, cap, floor and collar options are obtained.

The connection between spot and futures prices for commodities with restricted storability as well as the valuation of storage opportunities have attracted research interest for a long time. In this work, among others, the work [4], [10], [12], [25] and a general model in [22] are emphasized. Related to this work, the authors of [8] expose questions of pricing electricity and argue that the non-storability characteristics require a modeling of the production process. Another research direction, see [1], [13], [18], [5] and [15], focuses on modeling the stochastic dynamics of electrical spot prices, where the last three contributions also developed a risk-neutral point of view, which allows for valuing power derivatives. Pricing and hedging of spread options, with main focus on energy markets, is presented in [6].

The work is organized as follows. First a short overview of existing spot and futures price models will be given which will explain how this approach is related to them. Then, change-of numeraire techniques will be applied to derive consistent, risk neutral flow commodity price dynamics which are utilized to value cap, floor, collar and spread options. Thereafter, two approaches for calibrating the proposed model to historical futures prices are compared and the impact of parameter values on option prices is presented. We finish with some conclusions.

2 Modeling commodity price dynamics

In the last few years there has been a rapidly increasing volume of literature on stochastic models for electricity and other flow commodity prices. Basically, the models come in two varieties: either those suggesting a risk-neutral spot price dynamics or those describing the entire futures curve evolution. Here, an illustrative review of both approaches is given, focusing on the trackability with respect to valuation of derivative products.

Spot price models. The basic element in this class is an exogenously given *spot price* dynamics $(E_t)_{t\in[0,T]}$ of a flow commodity, which is described as an adapted process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$. Some authors (see e.g. [12], [25]) argue that a realistic approach has to respect the frequently observed mean-reverting behavior of flow commodity spot prices, which occurs due to restricted storability potentials. For instance, a popular choice for $(E_t)_{t\in[0,T]}$ is an Ornstein-Uhlenbeck-type process, whose dynamics is defined as solution to

$$dE_t = \kappa\,(b(t) - \log(E_t))\,E_t dt + \sigma E_t dW_t \qquad E_0 = E_0^* \in]0,\infty[. \qquad (1)$$

where E_0^* denotes the observed spot price at the beginning. Here κ, σ are positive constants and $b(\cdot)$ denotes a deterministic function, capturing the seasonal patterns in flow commodity prices. Moreover, $(W_t)_{t\in[0,T]}$ denotes a Brownian motion generating the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Note that the parameters of such a model are easily estimated (see e.g. [7], [18]). Having specified the spot price dynamics $(E_t)_{t\in[0,T]}$, the price evolution of a futures contract supplying one unit of commodity at time $\tau \in [0,T]$ is determined by

$$E_t(\tau) := E^Q[E_\tau \,|\, \mathcal{F}_t] \qquad \text{for all } t \in [0,\tau]$$

where Q is an appropriate martingale measure equivalent to P. Usually the measure Q is chosen (within a certain class) such that the observed initial futures curve $(E_0^*(\tau))_{\tau\in[0,T]}$ is explained as well as possible. The major drawback of the above model is the lack of flexibility to decouple spot and futures price evolutions. For these reasons authors of [24], [12] suggest extensions towards two sources of randomness. The work presented in [5] responds to these ideas proposing electricity spot prices in the form

$$E_t = \exp(b(t, l_t) + X_t + Y_t) \qquad t \in [0,T]. \qquad (2)$$

Here $(X_t)_{t\in[0,T]}$ is the so-called short term market fluctuation, $(Y_t)_{t\in[0,T]}$ is a process describing the long term dynamics and $b(\cdot,\cdot)$ is a function of the load forecast $(l_t)_{t\in[0,T]}$ including features of electricity production processes. This approach includes the impact of demand changes on electricity prices and captures seasonality as well as the mean reverting property of spot prices. The parameters of $(X_t)_{t\in[0,T]}$ are identified from historical data and $(Y_t)_{t\in[0,T]}$ is modeled such that observed futures prices are explained. Spot price models aim to capture spot and futures price behavior by exogenously given dynamics whose parameters fit most suitably to historical data.

Futures price models. The major disadvantage of the above price models is that the connection between spot and futures prices is too restrictive. More precisely, the modeled dynamics of the entire futures curve turns out to be rarely consistent with the actually observed curves. This shortcoming is at the core of the search for models that focus on observable properties of futures prices. In contrast to the above model class, futures price models attempt to systematically describe changes of the entire curve

$$\{E_t(\tau) : t, \tau \in [0, T], t \leq \tau\}.$$

By recognizing the fact that spot and futures prices coincide just in front of delivery, the spot price is recovered as $E_\tau = E_\tau(\tau)$ for $\tau \in [0, T]$.

A subclass of futures curve models is tracked in terms of the so-called *convenience yields*, a structure connecting spot and futures prices of a storable commodity. This approach is advocated in [21], [22] and extends the work of [4], [10], [12] and [25], where restricted storability is also considered. However, convenience yields argumentation relies on storability and is not sufficiently transferable to instantly perishable goods.

Another approach is based on the realization that excluding arbitrage for traded futures contracts means that their dynamics have to be adjusted in such a way that the price evolution $(E_t(\tau))_{t\in[0,\tau]}$ of each futures contract follows a martingale under some measure Q. Thus, several authors have proposed

$$\frac{dE_t(\tau)}{E_t(\tau)} = \sum_{k=1}^{n} \sigma_t^k(\tau)\ dW_t^k \qquad \text{for } t \in [0, \tau]. \tag{3}$$

Here $(W_t^1, \ldots, W_t^n)_{t\in[0,T]}$ denotes a n-dimensional standard Brownian motion and the volatilities are assumed to be deterministic functions $(t, \tau) \mapsto \sigma_t^k(\tau)$ of the current time t and the maturity τ for $k = 1, \ldots, n$. Such models yield explicit formulae for European calls written on futures prices, see [2], [6] and are further studied in [2], [7], [11], [17], [19] and [24]. For historical calibration of volatilities in (3), the principle component analysis (PCA) is proposed. This technique is based on the estimation of a covariance matrix and requires some stationarity in the volatility structure.

Futures versus spot price models. From the mathematical viewpoint, both model types are equivalent. Indeed, a futures price model establishes the futures price dynamics on $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t)_{t\in[0,T]})$ such that

$$(E_t(\tau))_{t\in[0,\tau]} \text{ is a martingale for each } \tau \in [0,T] \tag{4}$$

and commodity spot prices are given as terminal futures prices

$$E_t := E_t(t) \quad \text{for all } t \in [0,T]. \tag{5}$$

In contrast, spot price models construct on $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t)_{t\in[0,T]})$ the spot price as an

$$\text{adapted process } (E_t)_{t\in[0,T]} \tag{6}$$

from which futures prices are determined by conditional expectation

$$E_t(\tau) := E_Q(E_\tau \,|\, \mathcal{F}_t) \quad \text{for all } 0 \leq t \leq \tau \leq T. \tag{7}$$

This billateral correspondence highlights that any futures price model corresponds by (5) to an appropriate spot price model (6) and vice versa, any spot price model uniquely determines the corresponding futures model (4) due to (7). The quintessence is that it suffices to consider one of the model types. However, an appropriate model should fulfill a minimal set of requirements which will be discussed below.

2.1 Commodity prices under currency change

Suppose that for each $\tau \in [0,T]$ the futures price evolution $(E_t(\tau))_{t\in[0,\tau]}$ is a positive-valued adapted stochastic process realized on a complete filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$. It is also assumed that at the beginning $t = 0$ one observes prices $E^*_0(\tau)$ for all future delivery times $\tau \in [0,T]$ where the initial futures curve $(E^*_0(\tau))_{\tau\in[0,T]}$ is deterministic and continuous. It can be agreed that a reasonable model for a futures market obeys the following axioms.

C0 $(E_t(\tau))_{t\in[0,\tau]}$ is almost surely continuous for each $\tau \in [0,T]$.

C1 There is no arbitrage for $\{(E_t(\tau))_{t\in[0,\tau]} : \tau \in [0,T]\}$ in the sense that a risk-neutral measure Q^E equivalent to P exists such that for each $\tau \in [0,T]$, $(E_t(\tau))_{t\in[0,\tau]}$ follows a Q^E-martingale.

C2 Futures prices start at observed values $(E_0(\tau) = E_0^*(\tau))_{\tau\in[0,T]}$.

C3 Terminal prices $(E_t(t))_{t\in[0,T]}$ form a continuous (spot price) process.

Thus the explicit construction of a commodity market fulfilling the above requirements becomes essential. It turns out that a currency change provides a useful connection to fixed income markets. The idea is to express all futures prices in units of commodity prices just in front of delivery. In this new currency, commodity futures behave like zero bonds given by

$$p_t(\tau) := E_t(\tau)/E_t(t) \quad t \in [0,\tau],\ \tau \in [0,T]. \tag{8}$$

Money converts to a risky asset defined as

$$N_t := 1/E_t(t) \qquad t \in [0,T]. \tag{9}$$

Figure 1 illustrates the effect of this transformation on a typical realization of spot prices and two futures contract prices.

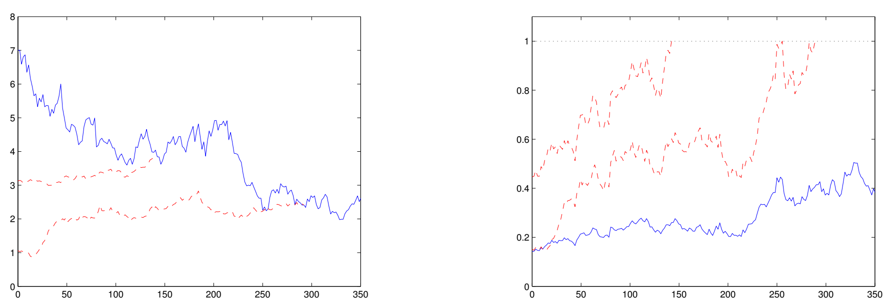

Figure 1: The dashed and solid lines on the left show typical realizations of commodity futures and spot prices respectively. The resulting bond and risky asset prices (8), (9) are displayed on the right.

Let us call a market consisting of zero bonds $\{p_t(\tau) : 0 \le t \le \tau \le T\}$ and an additional risky asset $(N_t)_{t\in[0,T]}$ *money market*. More precisely, such a market is characterized by axioms M0 to M3 below, which are mirrored from C0 to C3 by the above currency change (8), (9).

M0 $(N_t)_{t\in[0,T]}$, $(p_t(\tau))_{t\in[0,\tau]}$ are almost surely continuous for all $\tau \in [0,T]$.

M1 There is no arbitrage for $\{(N_t)_{t\in[0,T]}, (p_t(\tau))_{t\in[0,\tau]} : \tau \in [0,T]\}$ in the sense that a positive-valued adapted discounting process $(C_t)_{t\in[0,T]}$ exists as does a risk-neutral measure Q^M equivalent to P such that $(N_t/C_t)_{t\in[0,T]}$, $(p_t(\tau)/C_t)_{t\in[0,\tau]}$ are Q^M-martingales for all $\tau \in [0,T]$.

M2 Prices start at observed values $N_0 = N_0^*$, $(p_0(\tau) = p_0^*(\tau))_{\tau\in[0,T]}$.

M3 Bond prices finish at one $p_t(t) = 1$ for $t \in [0,T]$.

To make this text self-contained Theorem 1 proven in [14] is included below. It states that given a commodity market satisfying C0 to C3, the currency change (8), (9) yields a money market with M0 to M3. Moreover, the reverse currency change

$$E_t(\tau) := p_t(\tau)/N_t \qquad 0 \le t \le \tau \le T \tag{10}$$

applied on a money market yields a commodity market which obeys the axioms C0 to C3.

Theorem 1. *Let the set of chronological time pairs be denoted by* $\mathcal{D} := \{(t,\tau) : 0 \le t \le \tau \le T\}$.
(i) Suppose that the commodity market $(E_t(\tau))_{(t,\tau)\in\mathcal{D}}$ *fulfills C0 to C3 with initial futures curve* $(E_0^*(\tau))_{\tau\in[0,T]}$ *and risk-neutral measure* Q^E. *Then the transformation (8), (9) yields a money market satisfying M0 to M3 with initial values*

$$p_0^*(\tau) = E_0^*(\tau)/E_0^*(0) \quad \text{for all } \tau \in [0,T], \quad N_0^* = E_0^*(0)^{-1}$$

where the discounting process and the risk-neutral measure are given by

$$C_t = p_t(T) \quad \text{for all } t \in [0,T], \quad dQ^M = \frac{E_T(T)}{E_0(T)} dQ^E.$$

(ii) Suppose that the money market $(p_t(\tau))_{(t,\tau)\in\mathcal{D}}$, $(N_t)_{t\in[0,T]}$ *fulfills M0 to M3 with initial values* $(p_0^*(\tau))_{\tau\in[0,T]}$, N_0^*, *discounting process* $(C_t)_{t\in[0,T]}$, *and risk-neutral measure* Q^M. *Then the transformation (10) gives a commodity market with initial futures curve and risk-neutral measure*

$$E_0^*(\tau) = p_0^*(\tau)/N_0^* \quad \text{for all } \tau \in [0,T], \quad dQ^E = \frac{N_T}{C_T}\frac{C_0}{N_0} dQ^M.$$

Proof. (i) The properties M0, M1 and M3 are consequences of C0, C1, C3 due to (8) and (9). To prove M2, the change-of-numeraire technique is made of use (see e.g. [3]), which in this context applies as follows: For positive-valued adapted processes $(H_t)_{t\in[0,\tau]}$, $(D_t)_{t\in[0,T]}$ and $(D'_t)_{t\in[0,T]}$ holds

$$\left.\begin{array}{l}(H_t/D_t)_{t\in[0,\tau]} \text{ and } (D'_t/D_t)_{t\in[0,T]} \text{ are martingales with respect to} \\ Q \text{ if and only if } (H_t/D'_t)_{t\in[0,\tau]} \text{ and } (D_t/D'_t)_{t\in[0,T]} \text{ are martingales} \\ \text{with respect to } Q' \text{ given by } dQ' = \frac{D'_T}{D_T}\frac{D_0}{D'_0}dQ.\end{array}\right\} \quad (11)$$

Set now $H_t = E_t(\tau)$ for all $t \in [0,\tau]$ and $D_t = 1$, $D'_t = E_t(T)$ for all $t \in [0,T]$. Using (8), (9) and (11) it is concluded that

$$\left(\frac{E_t(\tau)}{E_t(T)} = \frac{p_t(\tau)}{p_t(T)}\right)_{t\in[0,\tau]} \quad \text{and} \quad \left(\frac{1}{E_t(T)} = \frac{N_t}{p_t(T)}\right)_{t\in[0,T]} \tag{12}$$

are martingales with respect to Q^M.
(ii) The properties C0, C1 and C3 are consequences of M0, M1 and M3 by (10). Finally, the measure Q^E is obtained from Q^M by using change of numeraire (11). □

Note that the introduced money market is merely a virtual object which is used to benefit from fixed income market theory. Namely, through (10) an *explicit construction* of commodity markets from well-established money market models is obtained. The most valuable feature of this approach is derived from the fact that common interest rate models easily enable changes in the shape of bond curves. Transferring this property by the reverse currency change (10) results in automatically obtaining commodity futures price models that incorporate the desired flexibility in futures curve evolution, for instance switches between market situations of backwardation and contango.

Another property of the presented methodology is the opportunity to model commodity markets with storable goods. It turns out that commodity storage costs correspond to restrictions on the short rate dynamics in money markets, where expressing all prices in a new currency one realizes that a self-financed strategy, which rolls its wealth over the next maturing futures behaves like a bank account paying a short rate, say $(r_t)_{t\in[0,T]}$. On the other hand, one considers a physical inventory where the storage costs are continuously covered by selling the commodity to the market at the spot price. This gives a decreasing rate of the inventory, say $(\kappa_t)_{t\in[0,T]}$. Now both, the wealth of futures rolling strategy and the inventory wealth play the role of savings with different interest rates $(r_t)_{t\in[0,T]}$ and $(\kappa_t)_{t\in[0,T]}$ respectively. Considering the restriction that $(\kappa_t)_{t\in[0,T]}$ applies exclusively to lending (since merely long positions in a physical storage are possible) whereas $(r_t)_{t\in[0,T]}$ applies to both, borrowing and lending, we obtain $r_t \geq \kappa_t$ for all $t \in [0,T]$, by no-arbitrage. Thus, to treat costly storable commodities, we have to investigate commodity markets based on those fixed income market models, where the short rate is bounded from below by an exogenously given inventory decay rate (which describes storage costs).

In what follows, commodity markets constructed from Gaussian Heath-Jarrow-Morton (HJM) methodology are considered. In this fixed income market model class, the short rate dynamic follows a Gaussian process and is not bounded from below. Hence, the obtained commodity futures price models override actual storage costs (supposing them to be infinite) and are thus applicable for instantly perishable commodities, such as electricity. Despite this property, it is suggested to use the same model also for other flow commodities, which are expensive to store, such as natural gas and oil.

Finally, it is emphasized that the currency change method provides a universal analogy between commodity and money markets in the sense that any money market (responding to M0 to M3) provides an appropriate commodity market (satisfying C0 to C3) and vice versa. As explained, the flexibility of the bond curves as well as restrictions on short rate dynamics in the money markets are essential in order to obtain realistic commodity market models.

At the current stage, the authors have to address some important details in the interest of future research. Among them is the question of which interest rate models provide reasonable commodity markets beyond the HJM approach presented below.

2.2 Market construction by HJM models

An explicit construction of flow commodity markets based on Gaussian Heath-Jarrow-Morton (HJM) interest rate models will now be illustrated. Let us begin with a complete filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$ where the filtration is the augmentation (by the null sets in $\mathcal{F}_T^W$) of the filtration $(\mathcal{F}_t^W)_{t\in[0,T]}$ generated by the d-dimensional Brownian motion $(W_t)_{t\in[0,T]}$. All processes are supposed to be progressively measurable. Assume that the observed, initial futures curve

$$(E_0^*(\tau))_{\tau\in[0,T]} \text{ is positive-valued, deterministic, absolutely continuous.} \quad (13)$$

Constructing the virtual money market, we specify the forward rate volatility $(\sigma_t(\tau))_{(t,\tau)\in\mathcal{D}}$ by choosing a deterministic function

$$\mathcal{D} \to \mathbb{R}^d,\ (t,\tau) \mapsto \sigma_t(\tau) \quad \text{with} \quad \int_0^T \int_0^\tau \|\sigma_t(\tau)\|^2 dt d\tau < \infty \quad (14)$$

and define bond volatilities by

$$\overline{\sigma}_t(\tau) := \int_t^\tau \sigma_t(s)ds \quad \text{for all } (t,\tau) \in \mathcal{D}. \tag{15}$$

Next the initial forward rates are determined

$$f_0^*(t) := -\frac{\partial}{\partial t} \log E_0^*(t) \quad \text{for all } t \in [0,T] \tag{16}$$

to define for all $(t,\tau) \in \mathcal{D}$ the forward rates as

$$f_t(\tau) := f_0^*(\tau) + \int_0^t \sigma_s(\tau)\overline{\sigma}_s(\tau)ds + \int_0^t \sigma_s(\tau)dW_s. \tag{17}$$

The bond price dynamics for all $\tau \in [0,T]$ are given by

$$dp_t(\tau) := p_t(\tau)(f_t(t)dt - \overline{\sigma}_t(\tau)dW_t), \qquad p_0(\tau) = p_0^*(\tau) := E_0^*(\tau)/E_0^*(0). \tag{18}$$

To construct an arbitrage-free money market, the evolution $(N_t)_{t\in[0,T]}$ of the additional risky asset is defined as

$$dN_t := N_t(f_t(t)dt + v_t dW_t) \qquad N_0 := E_0^*(0)^{-1}, \tag{19}$$

with a pre-specified d-dimensional deterministic volatility

$$(v_t)_{t\in[0,T]} \quad \text{with} \quad \int_0^T \|v_s\|^2 ds < \infty.$$

Moreover, the discounting process is given by

$$C_t := \exp\left(\int_0^t f_s(s)ds\right) \quad \text{for all } t \in [0,T]. \tag{20}$$

Define also

$$\Sigma_t(\tau) := -\overline{\sigma}_t(\tau) - v_t \quad \text{for all } (t,\tau) \in \mathcal{D}. \tag{21}$$

which is equal to the negative sum of the bond and asset price volatilities. Note that the initial values of forward rates, bond prices and the risky asset have to be chosen according to observations in the flow commodity market. Using standard results from interest rate theory, it can be verified that the above constructed money market can be transformed to a flow commodity market in the following way.

Theorem 2. *For* $(p_t(\tau))_{(t,\tau)\in\mathcal{D}}$, $(N_t)_{t\in[0,T]}$ *from (18), (19) define*

$$E_t(\tau) := p_t(\tau)/N_t \text{ for all } (t,\tau)\in\mathcal{D}.$$

Then $(E_t(\tau))_{(t,\tau)\in\mathcal{D}}$ *gives a commodity market with* $(E_0^*(\tau))_{\tau\in[0,T]}$ *from (13). Moreover, the risk-neutral measure satisfies*

$$dQ^E = \exp\left(\int_0^T v_s dW_s - \frac{1}{2}\int_0^T \|v_s\|^2 ds\right) dP$$

and futures prices follow

$$dE_t(\tau) = E_t(\tau)\Sigma_t(\tau)dW_t^E, \quad E_0(\tau) = E_0^*(\tau) \tag{22}$$

with Q^E*-Brownian motion*

$$W_t^E = -\int_0^t v_s ds + W_t, \qquad t\in[0,T] \tag{23}$$

Proof. Again, to make this approach self-contained, the proof from [14] is included. According to the previous theorem, it suffices to show that (18) and (19) define a money market with initial values $(p_0^*(\tau) = E_0^*(\tau)/E_0^*(0))_{\tau\in[0,T]}, N_0^* = E_0^*(0)^{-1}$, discounting process (20) and risk-neutral measure $Q^M = P$. The assumptions M0, M1 hold due to definition (18). Further, to see M3

$$p_t(\tau) = \exp\left(-\int_t^\tau f_t(s)ds\right) \qquad \text{for } (t,\tau)\in\mathcal{D},$$

is used (see Lemma 13.1.1, from [23]). Now, M2 is proven by verifying that $(N_t/C_t)_{t\in[0,T]}$ and $(p_t(\tau)/C_t)_{t\in[0,\tau]}$ are martingales with respect to $Q^M = P$. By Itô's formula, they admit stochastic differentials

$$d\left(\frac{N_t}{C_t}\right) = v_t\left(\frac{N_t}{C_t}\right)dW_t, \qquad d\left(\frac{p_t(\tau)}{C_t}\right) = -\overline{\sigma}_t(\tau)\left(\frac{p_t(\tau)}{C_t}\right)dW_t. \tag{24}$$

Next is proven for each $\tau\in[0,T]$, the process $(E_t(\tau))_{t\in[0,\tau]}$ follows a Q^E-martingale and by using (18), (19) and Itô's formula

$$\begin{aligned} dE_t(\tau) &= d\left(\frac{p_t(\tau)}{N_t}\right) = E_t(\tau)(-\Sigma_t(\tau)v_t dt + \Sigma_t(\tau)dW_t) \\ &= E_t(\tau)\Sigma_t(\tau)dW_t^E, \\ \text{with} \quad E_0(\tau) &= E_0^*(\tau). \end{aligned}$$

is found. Girsanov's theorem shows that (23) is in fact a Brownian motion under

$$dQ^E = \frac{N_T}{C_T}\frac{C_0}{N_0}dQ^M = \exp\left(\int_0^T v_s dW_s - \frac{1}{2}\int_0^T \|v_s\|^2 ds\right) dQ^M.$$

□

Risk neutral futures price dynamics based on Gaussian Heath-Jarrow-Morton interest rate models have explicitly been constructed. These dynamics not only take into account observed initial futures curves, allowing for backwardation as well as contango, but also reflect the non-storable nature of flow commodities. In the following the derived model will be applied for pricing frequently quoted energy derivatives.

3 Valuation of flow commodity options

Financial instruments enjoy an increasing popularity in commodity markets since they provide a custom-made protection against undesirable price movement for both, the producer and the consumer of the commodity. In the particular case of energy markets, options such as caps, floors and spreads are favored, as over a certain time horizon, they allow high fuel costs and/or low electricity prices to be hedged. Moreover, these contracts partially mimic a portfolio of real assets on energy production, which permits efficient option hedging by an appropriate dispatch of production units. In the following the financial instruments intended to be valued in the subsequent sections are defined.

Caps and floors. Caps and floors are options, which protect the buyer against high and low commodity prices over a fixed time horizon $[\tau_1, \tau_2]$. This is restricted to European type contracts, where during the time interval $[\tau_1, \tau_2]$ the buyer of a cap owns the right to receive a cash flow at intensity $((E_s(s) - K_c)^+)_{s\in[\tau_1,\tau_2]}$ with a price cap $K_c > 0$ previously specified in the contract. Similarly, a floor protects against low commodity prices within $[\tau_1, \tau_2]$ ensuring a cash flow at intensity $((K_f - E_s(s))^+)_{s\in[\tau_1,\tau_2]}$ with a price floor $K_f > 0$ at any time $s \in [\tau_1, \tau_2]$ of the contract. A combination of caps and floors, called collar, protects against high prices and foregoes returns from low prices giving the payoff intensity $((E_s(s)-K_c)^+ - (K_f - E_s(s))^+)_{s\in[\tau_1,\tau_2]}$. Thus, for $K_f < K_c$, the holder of a collar fixes own payment prices to the range $[K_f, K_c]$. It seems that energy producers prefer to sell collars where

$K_f < K_c$ are determined such that the initial collar price is zero. Therefore, no cash investment is required to reduce their potential loss from possible harmful prices.

For the reminder of this work, it is supposed that the riskless interest rate $r \in [0, \infty[$ is constant.

Proposition 1. *The fair price at time t of a cap option with price cap K_c written within the interval $[\tau_1, \tau_2]$ on commodity prices following (22) is*

$$\begin{aligned}
\mathrm{Cap}(t, K_c) &= \int_{t \vee \tau_1}^{\tau_2} e^{-r(\tau - t)} \left[E_t(\tau)\, \mathcal{N}(d_1(\tau)) - K_c\, \mathcal{N}(d_2(\tau)) \right] d\tau \qquad (25) \\
\text{with} \quad d_1(\tau) &= \frac{1}{D(\tau)} \left(\log\left(\frac{E_t(\tau)}{K_c} \right) + \frac{1}{2} D(\tau)^2 \right) \\
d_2(\tau) &= d_1(\tau) - D(\tau) \\
D(\tau)^2 &= \int_t^{\tau} \|\Sigma_u(\tau)\|^2 du \qquad (26)
\end{aligned}$$

Proof. According to the definition of a cap option, its fair price is

$$\mathrm{Cap}(t, K_c) = E^{Q^E} \left[\int_{t \vee \tau_1}^{\tau_2} e^{-r(\tau - t)} (E_\tau(\tau) - K_c)^+ d\tau \;\middle|\; \mathcal{F}_t \right]. \qquad (27)$$

Now, solve (22) by

$$E_\tau(\tau) = E_t(\tau) \exp\left(\int_t^{\tau} \Sigma_u(\tau) dW_u - \frac{1}{2} \int_t^{\tau} \|\Sigma_u(\tau)\|^2 du \right)$$

for each $\tau \in [\tau_1, \tau_2]$ and the result follows by straight-forward calculation. □

Note that the price of a cap option is actually equal to the integral

$$\mathrm{Cap}(t, K_c) = \int_{t \vee \tau_1}^{\tau_2} e^{-r(\tau - t)}\ \mathrm{BS}(E_t(\tau), K_c, \tau, t, 0, D(\tau)/\sqrt{\tau - t})\ d\tau \qquad (28)$$

of values obtained by the standard Black-Scholes formula

$$\begin{aligned}
\mathrm{BS}(s, k, \tau, t, r, \sigma) &:= s\mathcal{N}(d_+) - e^{-r(\tau - t)} k \mathcal{N}(d_-), \qquad (29) \\
d_+ &= \frac{1}{\sigma\sqrt{\tau - t}} \left[\log\left(\frac{s}{k} \right) + \left(r + \frac{1}{2}\sigma^2 \right) (\tau - t) \right], \\
d_- &= d_+ - \sigma\sqrt{\tau - t}.
\end{aligned}$$

Given the model parameters, we thus can use the Black-Scholes formula with *plug-in volatility*

$$\phi(\tau) \quad := \quad \frac{D(\tau)}{\sqrt{\tau - t}} \qquad \text{for } \tau \in [t \vee \tau_1, \tau_2]. \tag{30}$$

to calculate the integrand in (28).

In addition, the parity between cap and floor option prices is

$$\text{Cap}(t, K) - \text{Floor}(t, K) \quad = \quad \int_{t \vee \tau_1}^{\tau_2} e^{-r(\tau - t)} (E_t(\tau) - K) d\tau$$

showing that the cap is determined by the floor price and vice versa as $(E_t(\tau))_{\tau \in [t \vee \tau_1, \tau_2]}$ are observed at time t.

The fair value of a collar option follows by a combination of cap and floor prices

$$\text{Collar}(t, K_f, K_c) \quad = \quad \text{Cap}(t, K_c) - \text{Floor}(t, K_f).$$

Note that the fundamental component of cap, floor and collar option prices is the plug-in volatility $\phi(\cdot)$ given in (30). Next its form for a two-factor model is derived.

Example 1. Consider a two dimensional Gaussian HJM model specifying in (14) the constant and deterministic forward rate volatility as

$$\sigma_t(\tau) \quad := \quad [\sigma, 0] \qquad \text{for all } (t, \tau) \in \mathcal{D} \text{ with given } \sigma \in]0, \infty[. \tag{31}$$

Thus, bond volatilities are obtained from (15) as

$$\overline{\sigma}_t(\tau) \quad = \quad [\sigma(\tau - t), 0] \qquad \text{for all } (t, \tau) \in \mathcal{D}. \tag{32}$$

According to (19), the additional risky asset is determined by its volatility process $(v_t)_{t \in [0,T]}$. Suppose here a constant volatility, such that the additional asset dynamics admits a correlation to bond prices, i.e.

$$v_t \quad := \quad [v\rho, v\sqrt{1 - \rho^2}] \qquad \text{for all } t \in [0, T] \tag{33}$$

with a given constant volatility parameter $v \in]0, \infty[$ and a correlation parameter $\rho \in [-1, 1]$. Thus

$$\Sigma_t(\tau) \quad = \quad -\overline{\sigma}_t(\tau) - v_t = -[\sigma(\tau - t) + v\rho, v\sqrt{1 - \rho^2}]$$

is found and the plug-in volatility $\phi(\cdot)$ according to (30) can easily be calculated

$$\begin{aligned}\phi(\tau)^2 &= \frac{D(\tau)^2}{\tau - t} = \frac{1}{\tau - t}\int_t^\tau ||\Sigma_u(\tau)||^2 du \\ &= \frac{1}{\tau - t}\int_t^\tau (\sigma^2(\tau - u)^2 + 2v\rho\sigma(\tau - u) + v^2) du \\ &= \frac{1}{3}\sigma^2(\tau - t)^2 + v\rho\sigma(\tau - t) + v^2. \end{aligned} \tag{34}$$

Cap, floor, and collar options are priced by integrating values determined by the Black-Scholes formula (29) with plug-in volatility $\phi(\cdot)$ given by (34). In Section 4, it will be shown how to estimate the parameters σ, ρ and v from historical futures prices.

Cross commodity spreads. Naturally, a spread option is written on the price difference of two indexes, see e.g. [20], [6]. However, in commodity markets, the definition of the spread option has been relaxed resulting in several kind of contracts. Let us mention some of the most commonly traded products. Calendar and location spreads are written on the price of the same commodity at two different dates, respectively two different locations. In contrast, quality spreads are written on the price of different grades of the same commodity, often seen for oil products. Another class of spread options, written on the price difference of physical input and output commodities in a production process, is called processing spread. The most frequently quoted cross commodity spreads are spark spreads, which are written on power and fuel prices, crack spreads based on the price difference between crude and refined products and dark spreads written on the difference between electricity and coal prices.

Generally speaking, cross commodity options considered in this paper, are written on two different flow commodity spot prices $(G_t(t))_{t\in[0,T]}$, $(E_t(t))_{t\in[0,T]}$ over a fixed time horizon $[\tau_1, \tau_2]$ giving the owner the right to receive the cashflow

$$((\alpha G_\tau(\tau) - \beta E_\tau(\tau) - K)^+)_{\tau\in[\tau_1,\tau_2]}. \tag{35}$$

Here α and β are positive constants, which represent the number of contracts or the convertion ratios of involved flow commodities. Note that for spark spreads $\beta = 1$ and α refers to heat rate of the considered fuel. The constant K in (35) is called strike price. We derive fair prices of cross commodity spread options on two flow commodities with strike price $K = 0$ (other choices $K > 0$ do not possess explicit option formulae). Such spread options are called exchange options, because they allow the buyer to financially exchange one commodity for another without paying a strike price. Extending the modeling of flow commodity markets to the case of two simultaneously traded commodities, there is instead of (22), two dynamics

$$\begin{aligned} dG_s(\tau) &= G_s(\tau)\Sigma_s^G(\tau)dW_s^G, \qquad G_0(\tau) = G_0^*(\tau) &(36)\\ dE_s(\tau) &= E_s(\tau)\Sigma_s^E(\tau)dW_s^E, \qquad E_0(\tau) = E_0^*(\tau) &(37)\end{aligned}$$

for all $\tau \in [0,T]$. Here $(W_s^E)_{s\in[0,T]}$ and $(W_s^G)_{s\in[0,T]}$ are assumed to be standard Brownian motions with respect to $Q^{E,G}$ of dimensions $d_E, d_G \in \mathbb{N}$ with quadratic variation

$$d[W^{E,i}, W^{G,j}]_s = \gamma_{i,j} ds \quad i = 1,\dots,d_E,\ j = 1,\dots,d_G. \tag{38}$$

which is supposed to be

$$\Gamma := (\gamma_{i,j})_{i,j=1}^{d_E,d_G} \text{ constant and deterministic.} \tag{39}$$

Proposition 2. *Under the above assumptions, the fair price of a spread option determined by payoff intensity (35) with $K = 0$ is*

$$\mathrm{Spread}(t,\alpha,\beta) = \int_{t\vee\tau_1}^{\tau_2} e^{-r(\tau-t)} \left[\alpha G_t(\tau)\mathcal{N}(d_1(\tau)) - \beta E_t(\tau)\mathcal{N}(d_2(\tau))\right] d\tau \tag{40}$$

$$\begin{aligned} \textit{with}\quad d_1(\tau) &= \frac{1}{D^{E,G}(\tau)}\left[\log\left(\frac{\alpha G_t(\tau)}{\beta E_t(\tau)}\right) + \frac{1}{2}\, D^{E,G}(\tau)^2\right]\\ d_2(\tau) &= d_1(\tau) - D^{E,G}(\tau)\\ D^{E,G}(\tau)^2 &= \int_t^\tau \Sigma_u(\tau)^2 du\\ \Sigma_u(\tau)^2 &= \|\Sigma_u^G(\tau)\|^2 + \|\Sigma_u^E(\tau)\|^2 - 2\Sigma_u^E(\tau)\Gamma\Sigma_u^G(\tau)^\top. \end{aligned}$$

The proof is based on a simple verification that if a martingale possess a deterministic quadratic variation, then the corresponding stochastic exponential follows a lognormal distribution.

Lemma 1. *Let $(L_s)_{s\in[0,T]}$ be a continuous local martingale on a complete filtered probability space $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\in[0,T]})$ with deterministic quadratic variation and $L_0 = 0$. Then $(Z_s = Z_0\exp(L_s - \frac{1}{2}[L]_s))_{s\in[0,T]}$ satisfies*

$$E[(Z_\tau - K)^+ |\ \mathcal{F}_t] = Z_t N(d_1) - KN(d_2) \qquad \textit{for all } t \le \tau \tag{41}$$

with $d_1 = \frac{1}{\Sigma}\left(\log\left(\frac{Z_t}{K}\right) + \frac{1}{2}\Sigma^2\right),\ d_2 = d_1 - \Sigma,\ \Sigma^2 = [L]_\tau - [L]_t\,.$

Proof. Rewrite the left-hand side of (41) as

$$E\left[\left(Z_t \exp\left(L_\tau - L_t - \frac{1}{2}([L]_\tau - [L]_t)\right) - K\right)^+ \,\middle|\, \mathcal{F}_t\right]. \tag{42}$$

Using the deterministic time change

$$l(u) = \inf\{q \in [0,T] \,:\, [L]_q > u\} \quad \text{for all } u \in [0, [L]_T]$$

it is verified with time-change properties for continuous semimartingales (see [16], Theorem 4.6, Chapter 3) that

$$B_u := L_{l(u)}, \quad \mathcal{F}_u^B := \mathcal{F}_{l(u)} \qquad \text{for all } u \in [0,T]$$

defines a Brownian motion $(B_u, \mathcal{F}_u^B)_{u\in[0,T]}$, satisfying

$$L_u = B_{[L]_u} \quad \text{almost surely for all } u \in [0,T]. \tag{43}$$

Thus

$$L_\tau - L_t = B_{[L]_\tau} - B_{[L]_t} \tag{44}$$

follows centered Gaussian distribution with variance

$$\Sigma^2 = [L]_\tau - [L]_t.$$

Moreover, being an increment, (44) is independent from $\mathcal{F}^B_{[L]_t} = \mathcal{F}_{l([L]_t)}$ and also independent from $\mathcal{F}_t \subseteq \mathcal{F}_{l([L]_t)}$ where the inclusion holds due to $t \leq l([L]_t)$. The assertion follows now from (42) by a straight-forward derivation. □

The proof of Proposition 2 is now entered.

Proof. The expected payoff of a cross commodity spread is

$$\begin{aligned}
\text{Spread}(t,\alpha,\beta) &= E^{Q^{E,G}}\left[\int_{t\vee\tau_1}^{\tau_2} e^{-r(\tau-t)}(\alpha G_\tau(\tau) - \beta E_\tau(\tau))^+ d\tau \,\middle|\, \mathcal{F}_t\right] \\
&= \int_{t\vee\tau_1}^{\tau_2} e^{-r(\tau-t)} E^{Q^{E,G}}\left[(\alpha G_\tau(\tau) - \beta E_\tau(\tau))^+ \mid \mathcal{F}_t\right] d\tau \\
&= \int_{t\vee\tau_1}^{\tau_2} e^{-r(\tau-t)} E^{Q^{E,G}}\left[\alpha E_\tau(\tau)\left(\frac{G_\tau(\tau)}{E_\tau(\tau)} - \frac{\beta}{\alpha}\right)^+ \,\middle|\, \mathcal{F}_t\right] d\tau. \qquad (45)
\end{aligned}$$

To calculate the expectation within the last equation, the change-of-numeraire technique (11) is applied. Namely, for each $\tau \in [\tau_1 \vee t, \tau_2]$ a new measure Q^τ on $\mathcal{F}_\tau$ and a Q^τ-martingale $(Z_s(\tau))_{s\in[0,\tau]}$ are introduced

$$dQ^\tau = \frac{E_\tau(\tau)}{E_0(\tau)} dQ^{E,G}, \qquad Z_s(\tau) := \frac{G_s(\tau)}{E_s(\tau)}, \quad \text{for } s \in [0,\tau] \tag{46}$$

to transform the expectation in (45) to

$$E^{Q^\tau}\left[\left(Z_\tau(\tau) - \frac{\beta}{\alpha} \right)^+ \,\middle|\, \mathcal{F}_t \right] \alpha E_t(\tau) \; . \tag{47}$$

By applying Itô's formula, the stochastic differential

$$dZ_s(\tau) = Z_s(\tau) dL_s(\tau)$$

where for all $s \in [0,\tau]$

$$L_s(\tau) \;:=\; M_s^G(\tau) - M_s^E(\tau) + [M^E(\tau), M^G(\tau) - M^E(\tau)]_s \tag{48}$$

$$\text{with} \quad M_s^E(\tau) \;:=\; \int_0^s \Sigma_u^E(\tau) dW_u^E,$$

$$M_s^G(\tau) \;:=\; \int_0^s \Sigma_u^G(\tau) dW_u^G$$

is found. Hence, the quadratic variation of the local Q^τ-martingale $(L_s(\tau))_{s\in[0,\tau]}$ is deterministic

$$[L(\tau)]_s = \int_0^s \Sigma_u(\tau)^2 du \quad \text{for all } s \in [0,\tau]. \tag{49}$$

Lemma 1 yields immediately that (47) equals to

$$\alpha G_t(\tau) \mathcal{N}(d_1(\tau)) - \beta E_t(\tau) \mathcal{N}(d_2(\tau))$$

which is inserted into (45) to finish the proof. □

Summing up, it has been found that the price of a commodity spread option at time t equals to the integral over the time interval $[t \vee \tau_1, \tau_2]$ where the integrand is a European call option price multiplied by $e^{-r(\tau-t)}\alpha E_t(\tau)$, i.e.

$$\text{Spread}(t,\alpha,\beta) = \int_{t\vee\tau_1}^{\tau_2} e^{-r(\tau-t)} \; \alpha \; E_t(\tau) \; \text{BS}\left(\frac{G_t(\tau)}{E_t(\tau)}, \frac{\beta}{\alpha}, \tau, t, 0, \phi^{E,G}(\tau) \right) \; d\tau$$

with Black-Scholes formula (29) and plug-in volatility

$$\phi^{E,G}(\tau) := \frac{D^{E,G}(\tau)}{\sqrt{\tau - t}} \quad \text{for all } \tau \in [t \vee \tau_1, \tau_2].$$

Calendar spreads. Apart from cross commodity spread options, which protect the buyer against price differences between two commodities, calendar spreads are also of interest. Over time, these options hedge against price fluctuations of a single commodity and are of particular importance in commodity markets with restricted or even impossible storage. The most frequently quoted calendar spreads give the buyer at time t the right to sell one unit of commodity at τ_2 at a previously unknown price $E_{\tau_1}(\tau_1)$, where $t \leq \tau_1 \leq \tau_2$. This calender spread ensures at τ_2 the payoff

$$(E_{\tau_1}(\tau_1) - E_{\tau_2}(\tau_2))^+ \quad 0 \leq \tau_1 \leq \tau_2 \leq T. \tag{50}$$

Generally speaking, this option represents a European put with maturity τ_2 and strike price $E_{\tau_1}(\tau_1)$ which is undetermined at time $t < \tau_1$.

Proposition 3. *The fair price at time $t \in [0, \tau_2]$ of a calendar spread paying (50) at τ_2 written on a flow commodity with futures price dynamics (22) is given by*

$$\begin{aligned} \text{Calendar}(t, \tau_1, \tau_2) &= e^{-r(\tau_2 - t)} \left[E_{t\wedge\tau_1}(\tau_1)\ \mathcal{N}(d_1) - E_t(\tau_2)\ \mathcal{N}(d_2)\right] \\ \textit{with} \quad d_1 &= \frac{1}{D}\left(\log\left(\frac{E_{t\wedge\tau_1}(\tau_1)}{E_t(\tau_2)}\right) + \frac{1}{2}D^2\right) \\ d_2 &= d_1 - D \\ D^2 &= \int_t^{\tau_2} \|\Sigma_u(\tau_2) - \Sigma_{u\wedge\tau_1}(\tau_1) 1_{[0,\tau_1]}(u)\|^2 du. \end{aligned} \tag{51}$$

Proof. Obviously, the calendar spread price equals to

$$\text{Calendar}(t, \tau_1, \tau_2) = e^{-r(\tau_2 - t)} E^{Q^E} \left[(E_{\tau_1}(\tau_1) - E_{\tau_2}(\tau_2))^+ \mid \mathcal{F}_t\right]$$

Adapting the proof of Proposition 2 we set $\alpha = \beta = 1$ and replace $(G_s(\tau))_{s\in[0,\tau]}$, $(E_s(\tau))_{s\in[0,\tau]}$ by $(E_{s\wedge\tau_1}(\tau_1))_{s\in[0,\tau_2]}$ respectively $(E_s(\tau_2))_{s\in[0,\tau_2]}$. To obtain the assertion we then only have to insert on $[0, \tau_2]$ the dynamics

$$\begin{aligned} dE_{s\wedge\tau_1}(\tau_1) &= E_{s\wedge\tau_1}(\tau_1)\Sigma_{s\wedge\tau_1}(\tau_1) 1_{[0,\tau_1]}(s) dW_s^E \\ dE_s(\tau_2) &= E_s(\tau_2)\Sigma_s(\tau_2) dW_s^E. \end{aligned}$$

□

It is not surprising that the price of a calendar spread option is also related to the Black-Scholes formula. Note that the fundamental component in all derived pricing formulae is the plug-in volatility, whose term structure will be examined in the next section.

4 Historical model calibration

In this section, the calibration approach introduced in [15] (explicit estimation) for the two-factor model (31) - (33) is first reviewed. Thereafter, an alternative calibration procedure (log-likelihood estimation) based on a direct numerical maximization of the log-likelihood density is presented. This section concludes with a comparison of both methods, including an examination of the impact on option prices. In the sequel we restrict ourselves to the two-factor model (31) - (33).

The main issue of both approaches is the disentanglement of the relative price movements (fluctuation of fraction $E_t(\tau)/E_t(t)$ within $t \in [0, \tau]$) from the total movements (fluctuations of $E_t(\tau)$ within $t \in [0, \tau]$).

Explicit estimation. The idea is that with the construction (10) of commodity prices, we have

$$\frac{p_t(\tau_1)}{p_t(\tau_2)} = \frac{E_t(\tau_1)}{E_t(\tau_2)} \quad \text{for all } t \leq \tau_1 \leq \tau_2.$$

Thus, from the observations

$$\frac{E_{t_i}(\tau_1)}{E_{t_i}(\tau_2)}(\omega), \quad t_i \in \{t_0, \ldots, t_n\} \tag{52}$$

of the price fraction of two consecutive ($\tau_1 < \tau_2$) futures, information on parameters of the underlying money market can be extracted. Indeed, in Lemma 1 in [15], it is shown that the maximum-likelihood estimate of forward rate volatility σ from (31) based on (52) is

$$\hat{\sigma}^2 = \frac{-n + \sqrt{n^2 + 4\big(\sum_{i=0}^{n-1} b_i^2\big)\big(\sum_{i=0}^{n-1} a_i^2\big)}}{2\sum_{i=0}^{n-1} b_i^2} \tag{53}$$

where for $i = 0, \ldots, n-1$

$$a_i := \frac{\log\big(E_{t_{i+1}}(\tau_1)/E_{t_{i+1}}(\tau_2)\big) - \log\big(E_{t_i}(\tau_1)/E_{t_i}(\tau_2)\big)}{(\tau_2 - \tau_1)\sqrt{t_{i+1} - t_i}}(\omega), \tag{54}$$

$$b_i := \frac{(\tau_1 + \tau_2)(t_{i+1} - t_i) - (t_{i+1}^2 - t_i^2)}{2\sqrt{t_{i+1} - t_i}}. \tag{55}$$

The estimation of parameters v, ρ in (33) is more involving. Explicit estimates are merely obtained from time-equidistant observations. Namely, given σ and futures prices

$$E_{t_j}(\tau_1)(\omega), \quad E_{t_j}(\tau_2)(\omega) \quad j \in J := \{i \in \{0, \ldots, n-1\} : t_{i+1} - t_i = \delta\} \tag{56}$$

the maximum-likelihood estimate for the risky asset volatility v is given as

$$\hat{v}^2 = 2\frac{-|J| + \sqrt{|J|^2 + |J|\delta \sum_{j\in J}(c_j + \sigma^2\delta^{\frac{5}{2}}/24)^2}}{|J|\delta} - \frac{\sigma^2\delta^2}{12} \tag{57}$$

where for $i = 0, \ldots, n-1$

$$\begin{aligned} c_i \quad &:= \quad (t_{i+1} - t_i)^{-\frac{1}{2}}\Bigg[\log\left(\frac{E_{t_{i+1}}(\tau_1)}{E_{t_i}(\tau_1)}(\omega)\right) + \sigma^2\frac{(\tau_1 - t_i)^3 - (\tau_1 - t_{i+1})^3}{6} \\ &\qquad + d_i\frac{(\tau_1 - t_i)^2 - (\tau_1 - t_{i+1})^2}{2(t_{i+1} - t_i)}\Bigg] \end{aligned} \tag{58}$$

$$\begin{aligned} d_i \quad &:= \quad (\tau_2 - \tau_1)^{-1}\left[\log\left(\frac{E_{t_{i+1}}(\tau_1)}{E_{t_{i+1}}(\tau_2)}(\omega)\right) - \log\left(\frac{E_{t_i}(\tau_1)}{E_{t_i}(\tau_2)}(\omega)\right)\right] \\ &\qquad - \frac{\sigma^2}{2}\left[(\tau_1 + \tau_2)(t_{i+1} - t_i) - (t_{i+1}^2 - t_i^2)\right]. \end{aligned} \tag{59}$$

Given model parameters $\sigma \in]0, \infty[$ and $v \in]0, \infty[$, then Lemma 2 in [15] claims that ρ can be estimated by

$$\hat{\rho} = \frac{|J|^{-1}\sum_{j\in J} e_j d_j}{\sqrt{|J|^{-1}\sum_{j\in J}(e_j^2 - \frac{\sigma^2\delta^3}{12})}\sqrt{|J|^{-1}\sum_{j\in J} d_j^2}} \tag{60}$$

where $(d_i)_{i=0}^{n-1}$ are from (59) and

$$e_i \quad := \quad (t_{i+1} - t_i)\frac{v^2}{2} - (t_{i+1} - t_i)^{\frac{1}{2}} c_i. \qquad \text{for } i = 0, \ldots, n-1. \tag{61}$$

Summing up, formulae (53), (57), and (60) provide an estimation procedure for σ, v, and ρ based on observations (52) of two consecutive commodity futures prices.

Log-likelihood estimation. The above calculation of $\hat{\sigma}$ requires two consecutive futures, so the available historical price data must be reduced to those time points where both contracts are quoted. Moreover, the calculations of $\hat{v}$ and $\hat{\rho}$ require a reduction to equidistant time observations. To the contrary, a direct numerical log-likehood estimation would be based on a single futures price time series and thus could use all available data. Let us derive the likelihood function for direct estimation. Having observed at discrete times $t_0, \ldots, t_N \in [0, \tau]$ futures prices

$$E_{t_0}(\tau)(\omega),\ E_{t_1}(\tau)(\omega), \ldots, E_{t_N}(\tau)(\omega), \tag{62}$$

realizations of random variables

$$F_i \quad := \quad \log\left(\frac{E_{t_{i+1}}(\tau)}{E_{t_i}(\tau)}\right) \qquad i = 0, \ldots, N-1,$$

are obtained which, due to (22) are given as

$$F_i \quad = \quad \int_{t_i}^{t_{i+1}} -\frac{1}{2}\|\Sigma_s(\tau)\|^2 ds + \int_{t_i}^{t_{i+1}} \Sigma_s(\tau) dW_s^E. \tag{63}$$

Hence $(F_i)_{i=0}^{N-1}$ are normally distributed with respect to Q^E and the corresponding log-likelihood function $L_{f_0,\ldots,f_N}$ on the realizations $(f_0, \ldots, f_{N-1}) := (F_0, \ldots, F_{N-1})(\omega)$ fulfills

$$\begin{aligned} &L_{f_0,\ldots,f_N} :]0, \infty[\times]0, \infty[\times[-1, 1] \to \mathbb{R} \\ &(\sigma, v, \rho) \mapsto -\frac{N}{2}\log(2\pi) - \frac{1}{2}\sum_{i=0}^{N-1}\log(h_i(\sigma, v, \rho)) - \sum_{i=0}^{N-1}\frac{(f_i + \frac{1}{2}h_i(\sigma, v, \rho))^2}{2h_i(\sigma, v, \rho)} \end{aligned}$$

where for $i = 0, \ldots, N-1$

$$\begin{aligned} h_i(\sigma, v, \rho) \quad := \quad & \frac{1}{3}\sigma^2\left[(\tau - t_{i+1})^3 - (\tau - t_i)^3\right] \\ & -\sigma v \rho\left[(\tau - t_{i+1})^2 - (\tau - t_i)^2\right] + v^2(t_{i+1} - t_i). \end{aligned}$$

To maximize the function $L_{f_0,\ldots,f_N}$ the following procedure is applied: Having fixed two parameters, a line search algorithm determines the maximizer of the log-likelihood function in the remaining parameter. Then the same maximization is applied to each of the other parameters. This procedure is repeated until the maximal value changes less than a fixed tolerance. The initial parameter values are obtained from the explicit estimation described above.

Comparison of estimation methods. Let us comment on numerical results from both estimation procedures. The data used for calculations has been publicly available at `http://www.eex.de`, from the European Energy

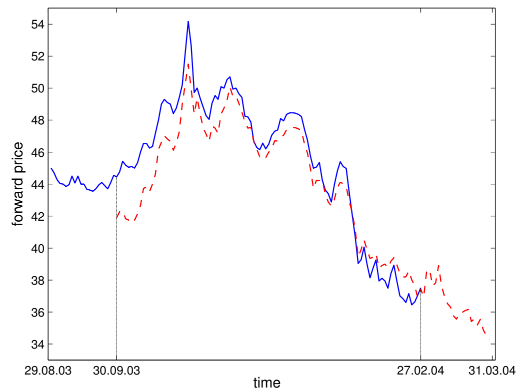

Figure 2: Daily prices for March 2004 and April 2004 monthly peak futures within common trading period from 30/9/2003 to 27/2/2004.

Exchange (EEX) in Leipzig, where next to power for physical delivery, financial forward and futures contracts are traded. For the estimations monthly peak load futures prices are examined. These derivatives deliver electricity within one month, daily from 8 a.m. to 8 p.m.. The futures are listed daily, six months prior to expiry, in EURO per MWh.

For the explicit estimation futures with successive delivery dates are considered, which results in a data series encompassing five months. Such data is illustrated in Figure 2 for $\tau_1 =$ 1st of March 2004, $\tau_2 =$ 1st of April 2004.

To compare the results, the numerical log-likelihood estimation has been run on exactly the same observations, i.e. using price data from only five months. The output of the explicit estimation based on contracts maturing at τ_1, τ_2 is denoted by $\hat{\sigma}^{\tau_2}_{\tau_1}$, $\hat{v}^{\tau_2}_{\tau_1}$, $\hat{\rho}^{\tau_2}_{\tau_1}$, whereas that of the log-likelihood estimation is denoted by $\hat{\sigma}_{\tau_1}$, $\hat{v}_{\tau_1}$, $\hat{\rho}_{\tau_1}$ for the preceding and $\hat{\sigma}^{\tau_2}$, $\hat{v}^{\tau_2}$, $\hat{\rho}^{\tau_2}$ for the subsequent contract in the time pair (τ_1, τ_2). Figure 3 depicts the points $(\hat{\sigma}^{\tau_2}_{\tau_1}, \hat{\sigma}_{\tau_1})$ (left) and points $(\hat{\sigma}^{\tau_2}_{\tau_1}, \hat{\sigma}^{\tau_2})$ (right) for τ_1 running through June 2003,..., May 2005. The same situation is illustraded by Figures 4 and 5 for the parameter estimates of v and ρ, respectively.

It is seen that in almost all cases outputs of the numerical log-likelihood procedure are lower than values obtained from explicit estimations. It seems interesting that $\hat{\rho}$ is always negative, frequently close to -1. Moreover, strong fluctuation of the values $\hat{\sigma}$ is observed.

A more intuitive way to interpret these estimates is to examine implications

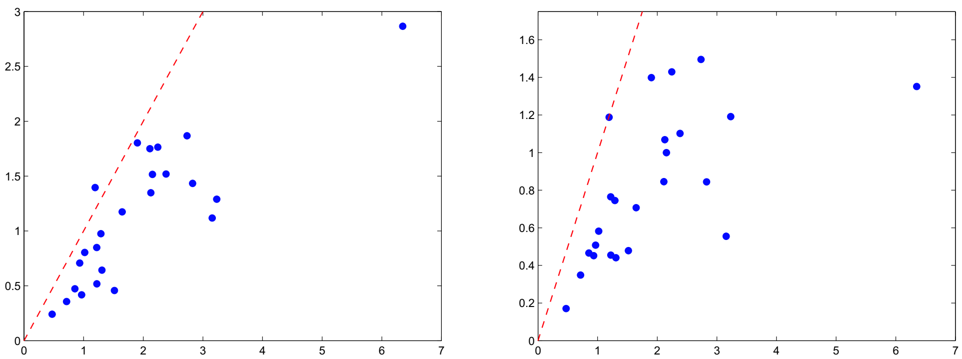

Figure 3: Forward rate volatility estimates $(\hat{\sigma}_{\tau_1}^{\tau_2}, \hat{\sigma}_{\tau_1})$ (left) and $(\hat{\sigma}_{\tau_1}^{\tau_2}, \hat{\sigma}^{\tau_2})$ (right) for $\tau_1 =$June 2003,..., May 2005 and $\tau_2 =$July 2003,...,June 2005.

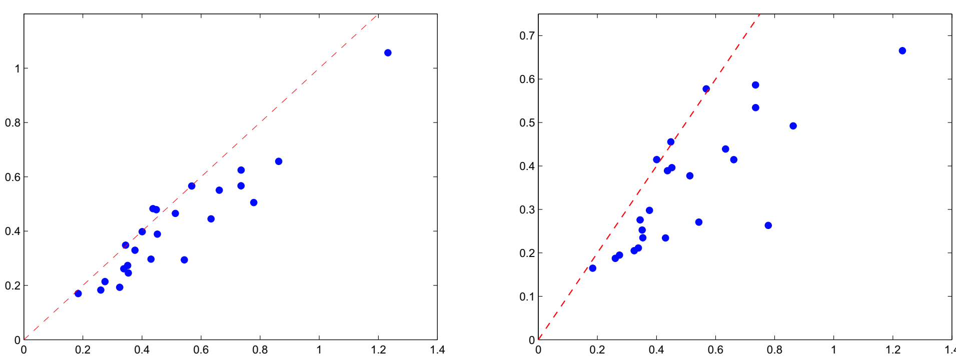

Figure 4: Spot volatility estimates $(\hat{v}_{\tau_1}^{\tau_2}, \hat{v}_{\tau_1})$ (left) and $(\hat{v}_{\tau_1}^{\tau_2}, \hat{v}^{\tau_2})$ (right) for $\tau_1 =$June 2003,..., May 2005 and $\tau_2 =$July 2003,...,June 2005.

for option prices. For example, consider the standard European call written on spot price giving at maturity τ the payoff $(E_\tau(\tau) - K)^+$. The price of this derivative at a fixed time $t \leq \tau$ is according to Proposition 1 given by

$$\mathrm{BS}(E_t(\tau), K, \tau, t, 0, \phi(\tau))$$

where the plug-in volatility (34) for this two-factor model is

$$\phi(\tau) = \sqrt{\sigma^2 \frac{(\tau - t)^2}{3} + 2\sigma\rho v \frac{(\tau - t)}{2} + v^2}\,. \tag{64}$$

The shape of the plug-in volatility is highly informative, since it illustrates how the flow commodity call option price differs from the same option price written on a storable asset. To investigate its form parameter estimates from data presented in Figure 2 are summarized in the following table.

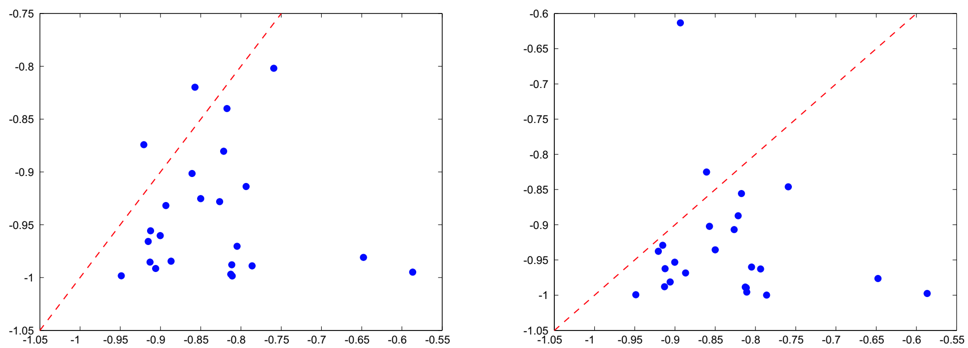

Figure 5: Correlation estimates $(\hat{\rho}^{\tau_2}_{\tau_1}, \hat{\rho}_{\tau_1})$ (left) and $(\hat{\rho}^{\tau_2}_{\tau_1}, \hat{\rho}^{\tau_2})$ (right) for $\tau_1 =$June 2003,..., May 2005 and $\tau_2 =$July 2003,...,June 2005.

Parameter	Explicit (τ_1, τ_2)	log-likelihood (τ_1)	log-likelihood (τ_2)
σ	1.9021	1.8030	1.3986
v	0.6338	0.4452	0.4389
ρ	-0.8215	-0.8804	-0.8872

Inserting these parameter values into (64) we find three different shapes displayed in Figure 6. An interesting result is that all three plug-in volatilities are increasing close to maturity. This confirms the statistical observations in [9], namely the Samuelson effect, which describes that derivatives in front of delivery exhibit a remarkable higher volatility than those further from maturity. From our standpoint, this fact affirms that the present approach yields a correct volatility term structure.

5 Conclusions

In this paper a new approach to pricing flow commodity derivatives has been followed. Starting from Gaussian Heath-Jarrow-Morton interest rate models, flow commodity markets have been canonically constructed. With futures price models described here, explicit formulae for caps, floors, collars, and cross commodity spreads are deduced by applying change-of-numeraire techniques. In addition, the derived option prices are connected to the classical Black-Scholes framework. The plug-in volatilities show correct qualitative term structures, namely the Samuelson effect, for both proposed calibration types.

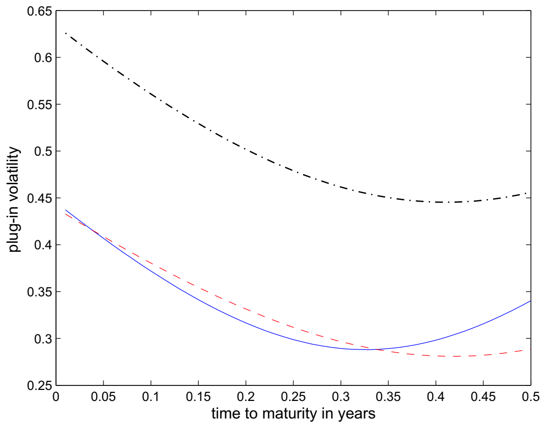

Figure 6: For $t = 0$, the plug-in volatility $(\phi(\tau))_{\tau \in [0, \frac{1}{2}]}$ is presented for different parameter estimates. The doted dashed, solid, and dashed line correspond to parameters in the 2nd, 3rd, and 4th column of the table.

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