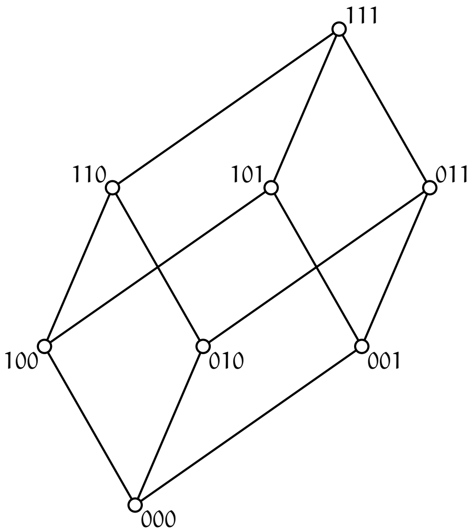
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

IFOR

Unique-Sink Orientations of Hypercubes and Linear Complementarity Problems

Jan Foniok
with Komei Fukuda, Bernd Gärtner, Lorenz Klaus, Hans-Jakob Lüthi

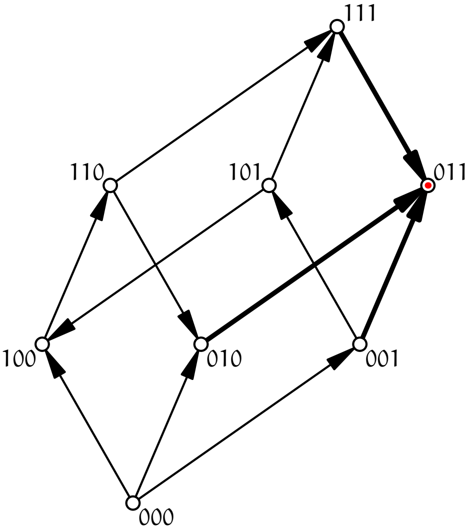
ODSA 2010



Unique-sink orientation — USO

an oriented graph with

- $V = \{0, 1\}^n$
- $u \sim v$ iff in Hamming distance 1

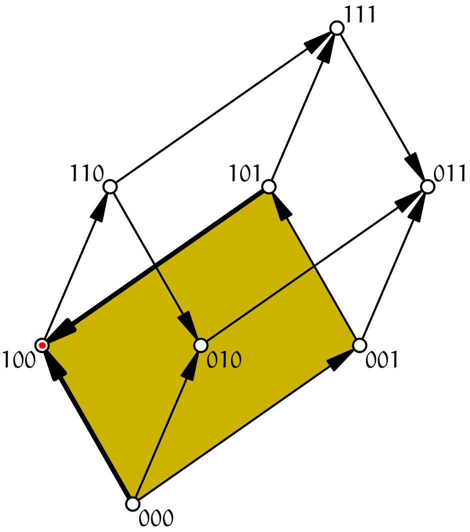


Unique-sink orientation — USO

an oriented graph with

- $V = \{0, 1\}^n$
- $u \sim v$ iff in Hamming distance 1
- oriented so that every subcube has a unique sink

So the whole cube must have a unique sink

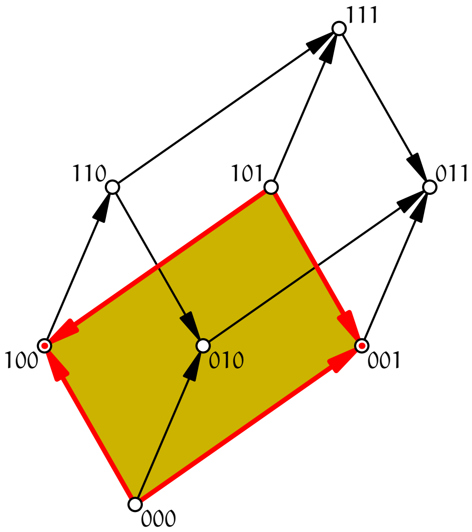


Unique-sink orientation — USO

an oriented graph with

- $V = \{0,1\}^n$
- $u \sim v$ iff in Hamming distance 1
- oriented so that every subcube has a unique sink

So the whole cube must have a unique sink, but also proper subcubes, like this square.

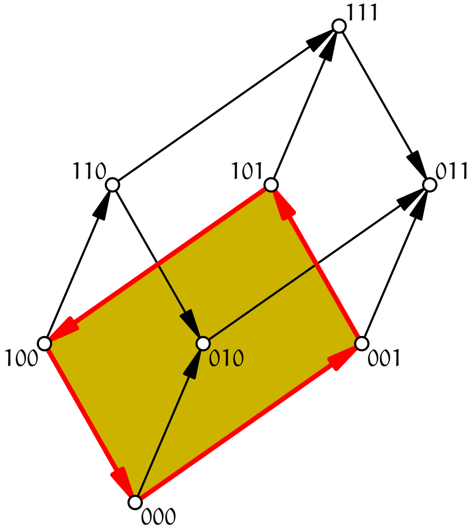


Unique-sink orientation — USO

an oriented graph with

- $V = \{0, 1\}^n$
- $u \sim v$ iff in Hamming distance 1
- oriented so that every subcube has a unique sink

So the whole cube must have a unique sink, but also proper subcubes, like this square. And not two sinks.

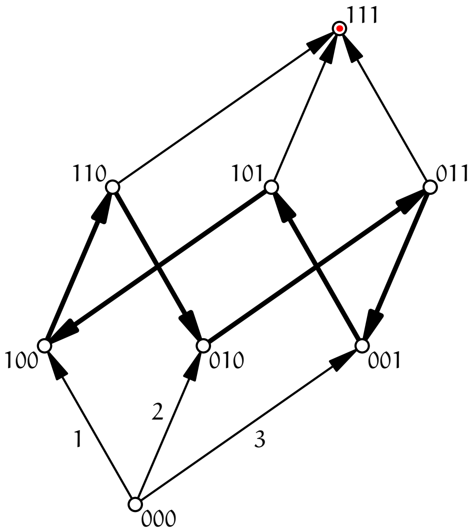


Unique-sink orientation — USO

an oriented graph with

- $V = \{0, 1\}^n$
- $u \sim v$ iff in Hamming distance 1
- oriented so that every subcube has a unique sink

So the whole cube must have a unique sink, but also proper subcubes, like this square. And not two sinks. And not none.



Unique-sink orientation — USO

an oriented graph with

- $V = \{0,1\}^n$
- $u \sim v$ iff in Hamming distance 1
- oriented so that every subcube has a unique sink

So the whole cube must have a unique sink, but also proper subcubes, like this square. And not two sinks. And not none. Cycles may occur.

Goal: Find the sink

Input representation: by the vertex enumeration oracle: ask for the orientation of edges incident with a given vertex

Algorithm efficiency: number of oracle calls as function of dimension

Algorithms

Naive algorithm: check all vertices (2^n queries)

Path-following algorithms: selection rule ??

"Random access" algorithms: seesaw

Best general algorithms known to date

	deterministic	randomized
general USOs	1.609^n Szabó, Welzl	1.438^n Szabó, Welzl, Rote
acyclic USOs		$\exp(2\sqrt{n})$ Matoušek, Sharir, Welzl, Gärtner

Why care?

Reductions from

- computing smallest enclosing ball of balls
- solving general linear and convex quadratic programs
- finding optimal strategies in simple stochastic games
- linear complementarity problems

Why care?

Reductions from

- linear complementarity problems

Linear Complementarity Problem (LCP)

Given

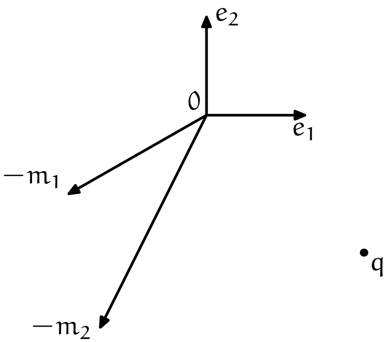
- an $n \times n$ real matrix M,
- a real n-dimensional vector q,

find

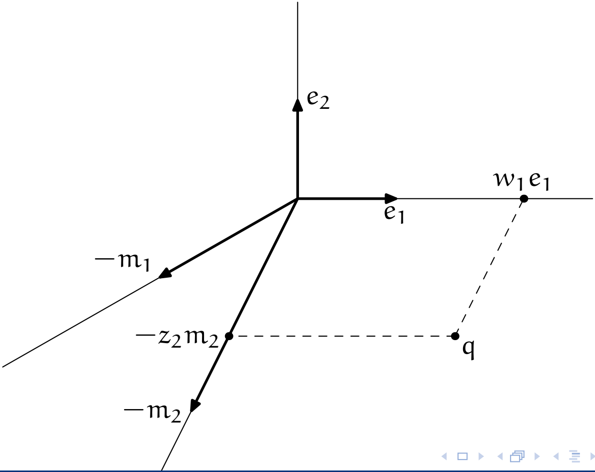
- two non-negative real n-dimensional vectors w, z such that

$$w - Mz = q$$
$$w^T z = 0$$

$$w - Mz = q$$
$$w^T z = 0$$



$$w - Mz = q$$
$$w^T z = 0$$



Complexity

NP-complete to decide whether a solution exists

Unsolved case: P-matrices (P-LCP)

A P-matrix is a matrix whose principal minors are all positive.

Is there a polynomial-time algorithm for solving P-LCP?

Theorem (Samelson, Thrall, Wesler 1958; Ingleton 1966)

A matrix M *is a P-matrix if and only if* $\mathrm{LCP}(M, q)$ *has a unique solution for every vector* q.

LCP Equations

$$q = w - Mz$$

$$w^T z = 0 \qquad \rightarrow \qquad w_i = 0 \text{ or } z_i = 0 \text{ for each } i$$

Problem reduction

The hard part: determine whether $w_i = 0$ or $z_i = 0$ for each i.

The rest is a system of linear equations.

LCP Equations

$$q = w - Mz$$
$$w^T z = 0 \qquad \rightarrow \qquad w_i = 0 \text{ or } z_i = 0 \text{ for each } i$$

Problem reduction

The hard part: determine whether $w_i = 0$ or $z_i = 0$ for each i.

The rest is a system of linear equations.

Inducing a USO

- a choice of $w_i = 0$ or $z_i = 0$ corresponds to a 0-1-vector
- solve equations: negative values $\Rightarrow$ outgoing edges
- *for a P-matrix, this is a USO* [Stickney, Watson, 1978]

Some matrix classes

P-matrix: all principal minors positive

K-matrix: P-matrix and all off-diagonal elements ≤ 0

Not all USOs can arise from a P-matrix LCP...

Some matrix classes

P-matrix: all principal minors positive

K-matrix: P-matrix and all off-diagonal elements ≤ 0

Not all USOs can arise from a P-matrix LCP...

Some USO classes

P-USO: coming from a P-matrix LCP

K-USO: coming from a K-matrix LCP

... in fact, very few of them do.

Theorem (F., Fukuda, Gärtner, Lüthi, 2009)

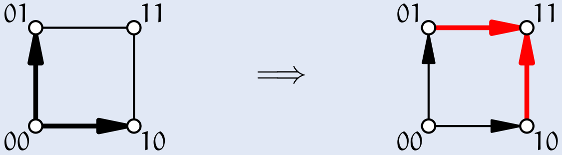
Any *path-following algorithm with* ***any*** *starting vertex finds the sink of any K-USO after at most* $2n+1$ *oracle queries.*

Theorem (F., Fukuda, Gärtner, Lüthi, 2009)

Any *path-following algorithm with* ***any*** *starting vertex finds the sink of any K-USO after at most* $2n + 1$ *oracle queries.*

Lemma

In any K-USO:



The proof uses a K-matrix characterization of [Fiedler, Pták, 1962], but can also be done purely combinatorially (oriented matroids) [F., Fukuda, Klaus, 2010].

Does the “Lemma” characterize K-USOs?

No. Because:

There are at least $2^{2^{n/\operatorname{poly}(n)}}$ n-dimensional USOs satisfying the “Lemma”, but at most $2^{O(n^3)}$ P-USOs.

Does the “Lemma” characterize K-USOs?

No. Because:

There are at least $2^{2^{n/\operatorname{poly}(n)}}$ n-dimensional USOs satisfying the “Lemma”, but at most $2^{O(n^3)}$ P-USOs.

Proof of the upper bound (F., Gärtner, Klaus).

The orientation is determined by the signs of $2^n \cdot n$ values of polynomials in the entries of M and q. Each of the polynomials has degree at most n.

Theorem (Warren, 1968)

The number of distinct (nowhere-zero) sign patterns of s real polynomials in k variables, each of degree at most d, is at most $(4eds/k)^k$.

Counting USOs

class	lower bound	upper bound
all USOs [Matoušek]	$n^{\Omega(2^n)}$	$n^{O(2^n)}$
acyclic USOs [Matoušek]	$2^{2^{n-1}}$	$(n+1)^{2^n}$
satisfying “Lemma”	$2^{2^n/\sqrt{n}}$	
Holt–Klee USOs [Develin]	$2^{2^n/\,\mathrm{poly}(n)}$	
P-USOs	$2^{\Omega(n^2)}$	$2^{O(n^3)}$
K-USOs	$2^{\Omega(n)}$	

Counting USOs

class	lower bound	upper bound
all USOs [Matoušek]	$n^{\Omega(2^n)}$	$n^{O(2^n)}$
acyclic USOs [Matoušek]	$2^{2^{n-1}}$	$(n+1)^{2^n}$
satisfying “Lemma”	$2^{2^n/\sqrt{n}}$	
Holt–Klee USOs [Develin]	$2^{2^n/\operatorname{poly}(n)}$	
P-USOs	$2^{\Omega(n^2)}$	$2^{O(n^3)}$
K-USOs	$2^{\Omega(n)}$	

Holt–Klee USOs

In every subcube of dimension d there are d vertex-disjoint directed paths from the (unique) source to the (unique) sink.

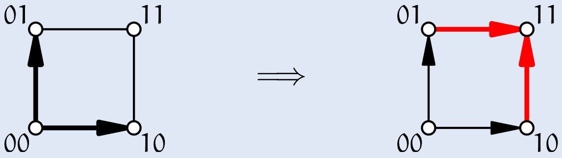
Every P-USO is Holt–Klee. [Gärtner, Morris, Rüst, 2008]

Counting USOs

class	lower bound	upper bound
satisfying "Lemma"	$2^{\binom{n-1}{\lfloor (n-1)/2 \rfloor}}$	

Lemma

In any K-USO:



P-USOs: A recursive construction

The P-matrix:

$$M = \begin{pmatrix} M' & 0 \\ m^T & 1 \end{pmatrix}$$

$$m_j = \begin{cases} (q_n + 1)M'_{jj}/q'_j & \text{if } s_j = -, \\ (q_n - 1)M'_{jj}/q'_j & \text{if } s_j = +, \end{cases}$$

$$q_n = \begin{cases} -1 & \text{if } s_n = -, \\ 1 & \text{if } s_n = +; \end{cases}$$

Every choice $s \in \{-, +\}^n$ yields a different P-USO.

Conclusion: Why do I think it's interesting?

- interplay of several areas of mathematics
  - linear algebra & (continuous) geometry
  - discrete geometry
  - algebraic geometry
  - combinatorics & order theory
- embarrassingly open complexity status
- strongly polynomial algorithm for linear programming ?!?

Some open problems

- complexity: Is P-matrix LCP PPAD-complete?
- counting: The number of USOs for a single P-matrix: $\Omega(2^{n^2})$?