The Linear Complementarity Problem: A Combinatorial Approach

The linear complementarity problem

The *linear complementarity problem (LCP)* for a given n by n real matrix M and a real n-vector q is to find non-negative vectors w and z such that

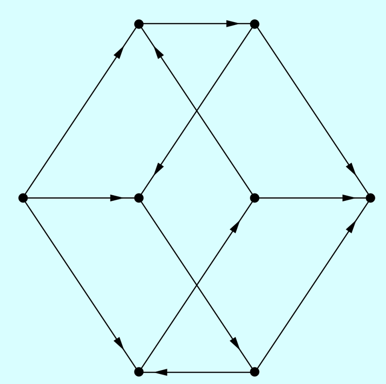
$$w - Mz = q$$
$$w^T z = 0.$$

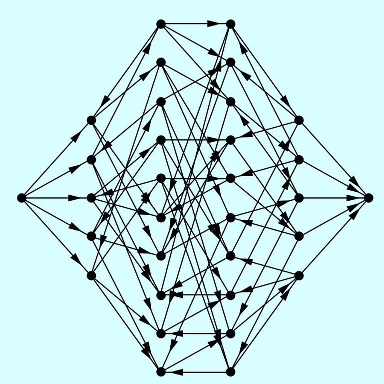
Together with non-negativity of w and z, the second condition implies that for every i we have $w_i = 0$ or $z_i = 0$.

To solve the LCP, one has to find out for which values of i the corresponding z_i should be zero or non-zero. To this end, we use *simple principal pivot algorithms*.

Unique-sink orientations of hypercubes

In the case of P-matrices, a step of the algorithm (admissible pivot) corresponds to an edge transversal in the n-dimensional cube. Admissible pivots induce an orientation of the hypercube.





Such an induced orientation is a *unique-sink orientation*: each face of the cube has exactly one sink. Solving the LCP is equivalent to finding the sink of the cube.

Randomised algorithms

We consider three randomised algorithms for finding the sink of a unique-sink orientation. Each of them follows a directed path from a given starting vertex to the sink.

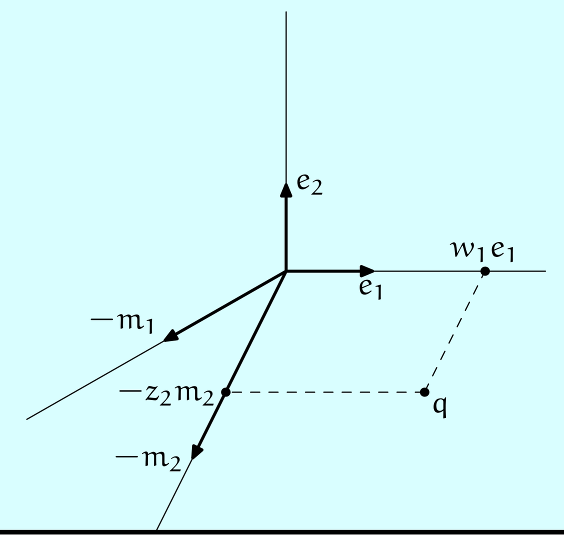
- *random edge* follows an outgoing edge selected at random in each step
- *random facet* randomly selects a facet adjacent to the current vertex and solves it recursively
- *randomised Murty's method* or *random permutation* follows the outgoing edge with the least index in an ordering chosen randomly in the beginning

Random edge is known to perform badly on some P-LCP orientations. No such negative result is currently known for the other two algorithms.

Motivations

LCP is a unifying formulation for *linear programming, convex quadratic programming, bimatrix games*.

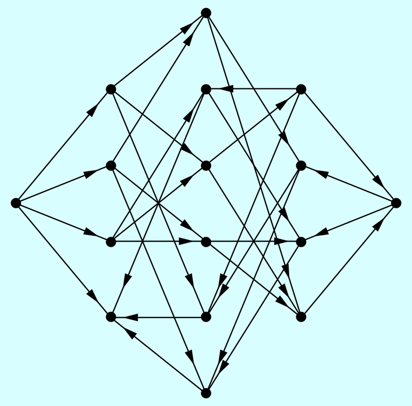
Particularly interesting is the case when M is a *P-matrix* (all principal minors are positive). A polynomial-time algorithm for P-LCP exists unless NP = co-NP, but no such algorithm is known.



The K-matrix case

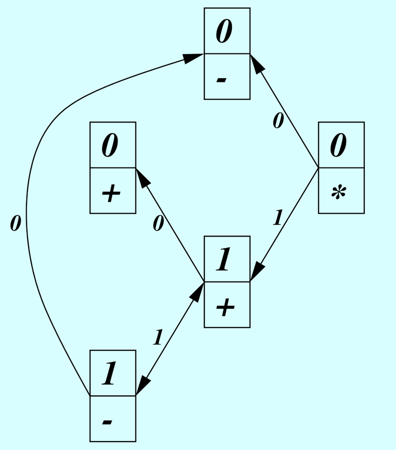
A *K-matrix* is a real matrix with non-positive off-diagonal entries and positive principal minors. K-matrix orientations satisfy nice combinatorial properties, which allow to prove the following.

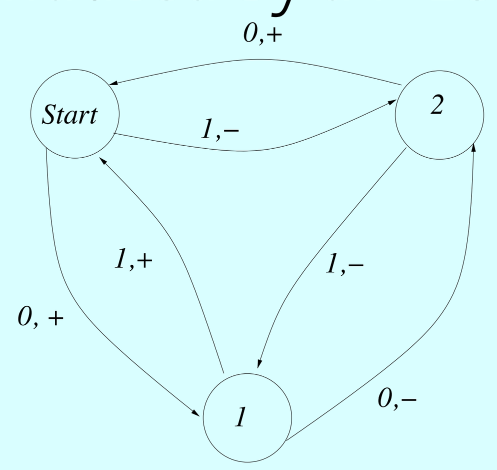
Theorem (2008). A simple principal pivot algorithm with an *arbitrary pivot rule* finds the solution to a K-matrix LCP after at most n steps.



Finite state transducers

Some families of unique-sink orientations can be simply described using a *finite state transducer*. We use this description to show that *random permutation* is fast on W.D. Morris's example, on which random edge is slow. Moreover, we find an explicit example of orientations on which *random permutation* is slow. However, the latter unique-sink orientations are not induced by a P-LCP.





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