The Linear Complementarity Problem: A Combinatorial Approach

The Linear Complementarity Problem (LCP) for a given \( n \times n \) real matrix \( M \) and a real \( n \)-vector \( q \) is to find non-negative vectors \( w \) and \( z \) such that

\[
\begin{align*}
    w - Mz &= q \\
    w^T z &= 0.
\end{align*}
\]

Together with non-negativity of \( w \) and \( z \), the second condition implies that for every \( i \) we have \( w_i = 0 \) or \( z_i = 0 \).

To solve the LCP, one has to find out for which values of \( i \) the corresponding \( z_i \) should be zero or non-zero. To this end, we use \textit{simple principal pivot algorithms}.

**Motivations**

LCP is a unifying formulation for \textit{linear programming}, \textit{convex quadratic programming}, \textit{bimatrix games}.

Particularly interesting is the case when \( M \) is a \textit{P-matrix} (all principal minors are positive). A polynomial-time algorithm for P-LCP exists unless \( \text{NP} = \text{co-NP} \), but no such algorithm is known.

**The K-matrix case**

A \textit{K-matrix} is a real matrix with non-positive off-diagonal entries and positive principal minors. K-matrix orientations satisfy nice combinatorial properties, which allow to prove the following.

**Theorem (2008).** A simple principal pivot algorithm with an \textit{arbitrary pivot rule} finds the solution to a K-matrix LCP after at most \( n \) steps.

**Unique-sink orientations of hypercubes**

In the case of P-matrices, a step of the algorithm (admissible pivot) corresponds to an edge traversal in the \( n \)-dimensional cube. Admissible pivots induce an orientation of the hypercube.

Such an induced orientation is a \textit{unique-sink orientation}: each face of the cube has exactly one sink. Solving the LCP is equivalent to finding the sink of the cube.

**Randomised algorithms**

We consider three randomised algorithms for finding the sink of a unique-sink orientation. Each of them follows a directed path from a given starting vertex to the sink.

- \textit{random edge} follows an outgoing edge selected at random in each step
- \textit{random facet} randomly selects a facet adjacent to the current vertex and solves it recursively
- \textit{randomised Murty’s method} or \textit{random permutation} follows the outgoing edge with the least index in an ordering chosen randomly in the beginning

\textit{Random edge} is known to perform badly on some P-LCP orientations. No such negative result is currently known for the other two algorithms.

**Finite state transducers**

Some families of unique-sink orientations can be simply described using a \textit{finite state transducer}. We use this description to show that \textit{random permutation} is fast on W.D. Morris’s example, on which random edge is slow. Moreover, we find an explicit example of orientations on which \textit{random permutation} is slow. However, the latter unique-sink orientations are not induced by a P-LCP.

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