Polynomial Pivoting for K-LCP
A Proof Using Unique-Sink Orientations

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Linear Complementarity Problem (LCP)

Given

- an $n \times n$ real matrix $M$,
- a real $n$-dimensional vector $q$,

find

- two non-negative real $n$-dimensional vectors $w, z$ such that

\[ w - Mz = q \]
\[ w^Tz = 0 \]
Applications

- quadratic programming
- bimatrix games
- control problems (parametric LCP)
Complexity

In general, LCP is NP-hard; it is NP-complete to decide whether a solution exists.
\[-m_2\]

\[-m_1\]
\[-m_2\]
\[-m_1\]
$e_1 \neq e_2 - m_2 - m_1$
Complexity

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Unsolved case: P-matrices (P-LCP)

A P-matrix is a matrix whose principal minors are all positive.

Is there a polynomial-time algorithm for solving P-LCP?
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Theorem (Samelson, Thrall, Wesler 1958; Ingleton 1966)

*A matrix $M$ is a P-matrix if and only if $\text{LCP}(M, q)$ has a unique solution for every vector $q$.**
\[ e_1 - e_2 - m_1 - m_2 \]
LCP Equations

\[ q = w - Mz \]
\[ w^Tz = 0 \]

Problem reduction

The hard part: determine whether \( w_i = 0 \) or \( z_i = 0 \) for each \( i \).

The rest is a system of linear equations.
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Simple principal pivoting methods

- start with an arbitrary complementary basis
- if not feasible, do a principal pivot:
  - insert a (negative) variable into the basis (pivot rule!)
  - remove the complementary variable from the basis
- repeat until solution is reached
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Goal
A polynomial number of principal pivots (iterations).
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Cube orientations

- represent each complementary basis with an \( n \)-dimensional 0, 1-vector
- 0, 1-vectors are vertices of the \( n \)-dimensional cube
- orient the edges in the direction of potential pivot steps
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## Theorem (Stickney, Watson 1978)

*The induced cube orientation for a non-degenerate P-LCP problem is a unique-sink orientation.*
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Theorem (Stickney, Watson 1978)

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Unique-sink orientation (USO)

An orientation of the \( n \)-dimensional cube is a unique-sink orientation if every subcube has exactly one sink.
### K-matrices

A **K-matrix** is a P-matrix whose off-diagonal elements are all non-positive. (K-LCP, K-USO)

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**Theorem (JF, Fukuda, Gärtner, Lüthi 2008)**

In a K-USO:

- There are no directed cycles.
- Every directed path from the sink to the sink has length at most $n$.
- Every directed path has length at most $2n$.

**Corollary**

The simple principal pivoting method with any pivot rule solves K-LCP in at most $2n$ iterations.
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### Deterministic vs. randomised pivot rules

- There tends to be a “bad example” for any studied deterministic pivot rule.
- Therefore examine **randomised pivot rules**, analyse **expected** running time.
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Open problems

- Is there a polynomial-time algorithm for P-LCP?
- Is there a deterministic pivot rule with which simple principle pivoting needs a polynomial number of iterations?
- Is there a randomised pivot rule with a polynomial expected number of iterations?
- In particular, is \textsc{Random Permutation} such a rule?