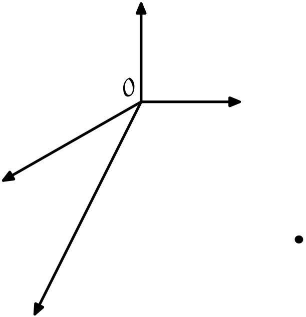
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Polynomial Pivoting for K-LCP
A Proof Using Unique-Sink Orientations

Jan Foniok
joint work with Komei Fukuda, Bernd Gärtner, Hans-Jakob Lüthi

Sixth Joint Operations Research Days, Lausanne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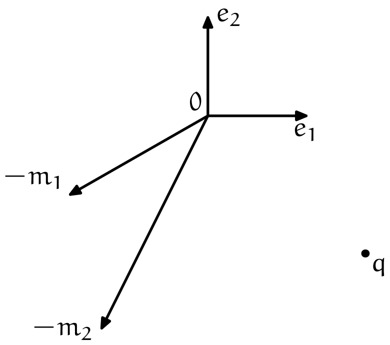
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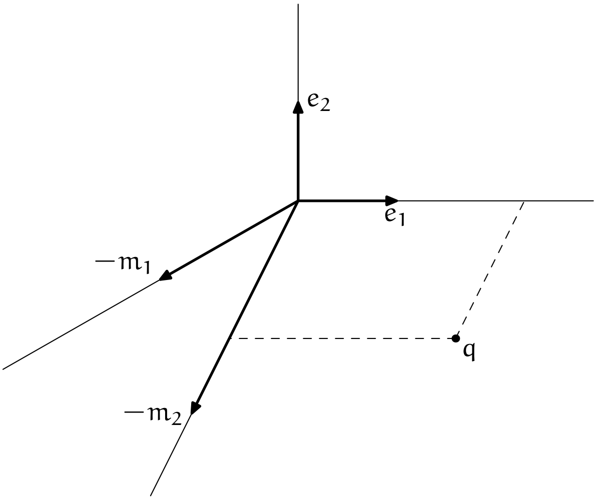
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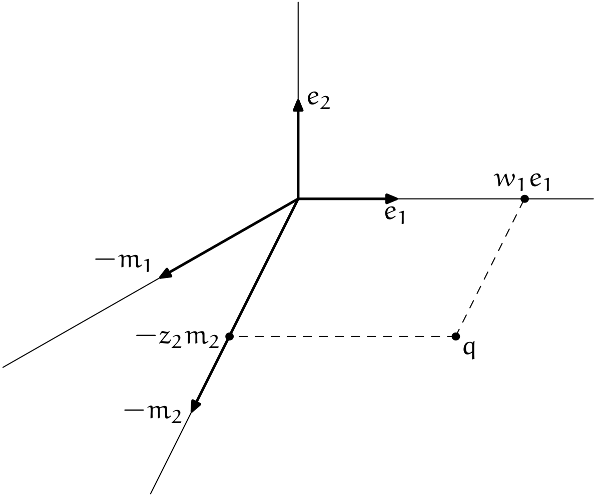

e_2
$w_1 e_1$
e_1
$-m_1$
$-z_2 m_2$
q
$-m_2$

Linear Complementarity Problem (LCP)

Given

- an $n \times n$ real matrix M,
- a real n-dimensional vector q,

find

- two non-negative real n-dimensional vectors w, z such that

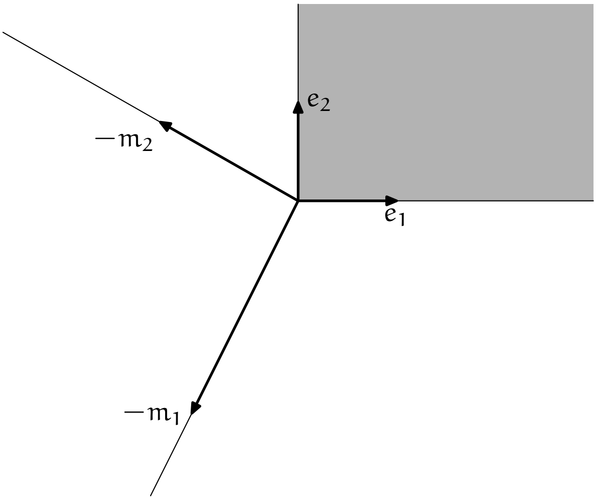
$$
\begin{aligned}
w - Mz &= q \\
w^T z &= 0
\end{aligned}
$$

Applications

- quadratic programming
- bimatrix games
- control problems (parametric LCP)

Complexity

In general, LCP is NP-hard; it is NP-complete to decide whether a solution exists.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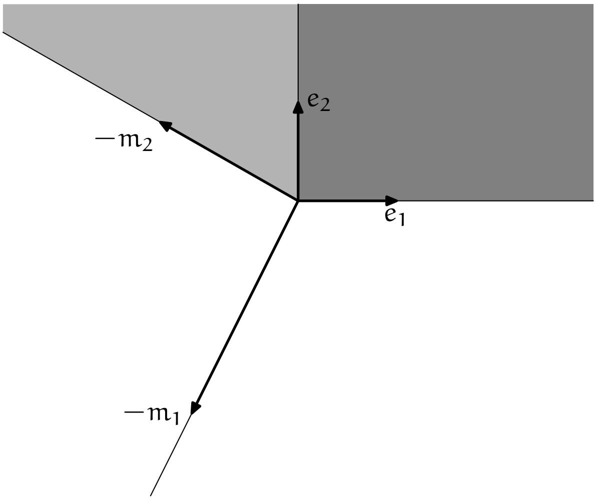
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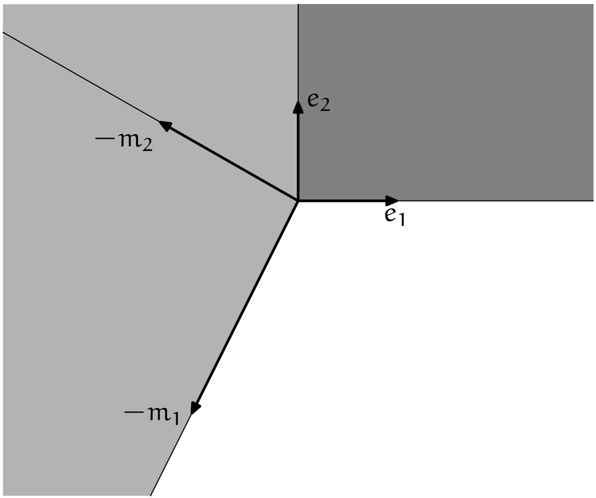
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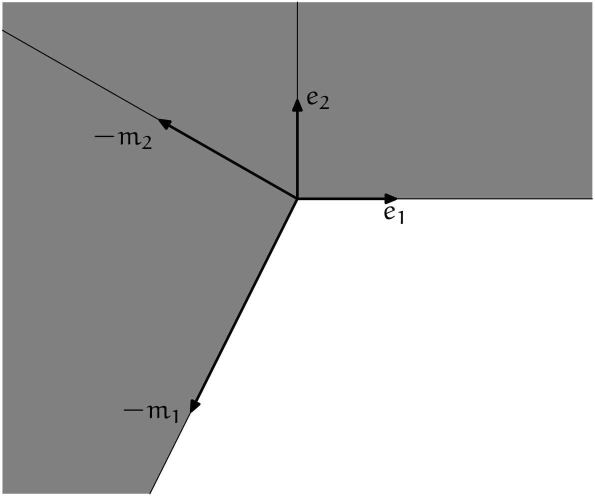
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Complexity

In general, LCP is NP-hard; it is NP-complete to decide whether a solution exists.

Unsolved case: P-matrices (P-LCP)

A P-matrix is a matrix whose principal minors are all positive.

Is there a polynomial-time algorithm for solving P-LCP?

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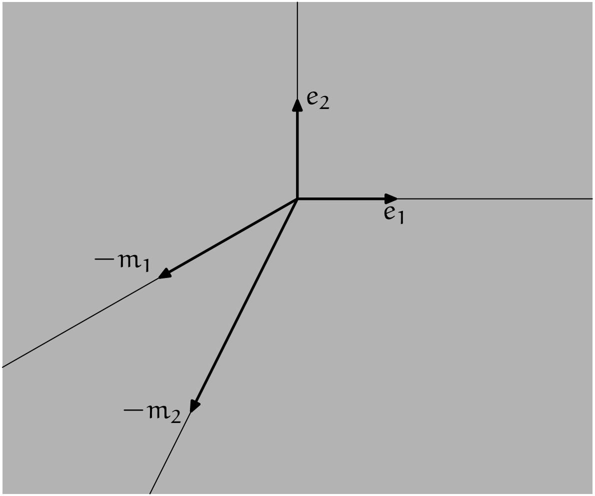
Theorem (Samelson, Thrall, Wesler 1958; Ingleton 1966)

A matrix M is a P-matrix if and only if $\mathrm{LCP}(M, q)$ *has a unique solution for every vector* q.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e_1
$-m_1$
$-m_2$

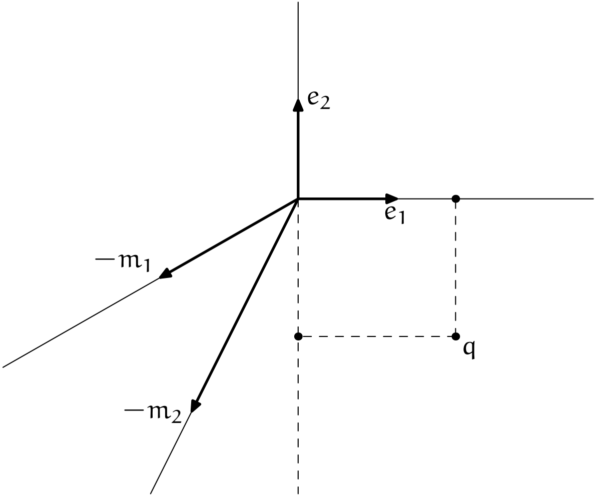
LCP Equations

$$q = w - Mz$$
$$w^T z = 0$$

Problem reduction

The hard part: determine whether $w_i = 0$ or $z_i = 0$ for each i.

The rest is a system of linear equations.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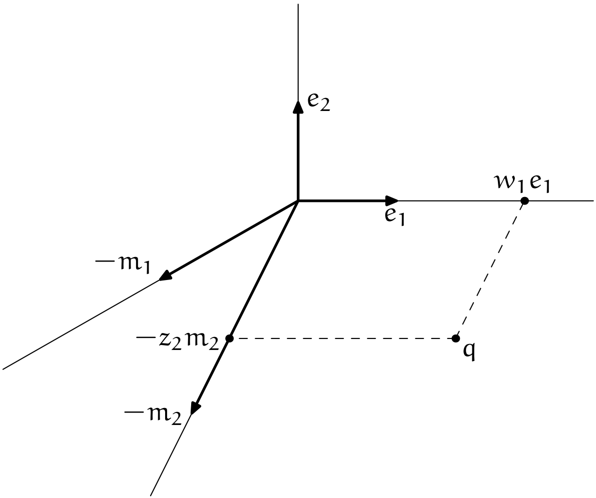
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Simple principal pivoting methods

- start with an arbitrary complementary basis
- if not feasible, do a principal pivot:
  - insert a (negative) variable into the basis (pivot rule!)
  - remove the complementary variable from the basis
- repeat until solution is reached

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Goal

A polynomial number of principal pivots (iterations).

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Cube orientations

- represent each complementary basis with an n-dimensional $0, 1$-vector
- $0, 1$-vectors are vertices of the n-dimensional cube
- orient the edges in the direction of potential pivot steps

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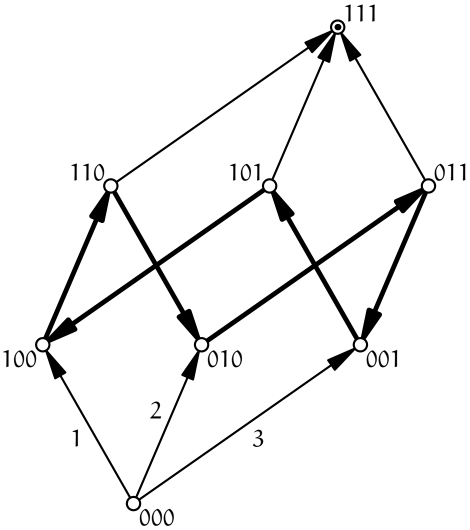
Unique-sink orientation (USO)

An orientation of the n-dimensional cube is a unique-sink orientation if every subcube has exactly one sink.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2
3

K-matrices

A K-matrix is a P-matrix whose off-diagonal elements are all non-positive. (K-LCP, K-USO)

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Theorem (JF, Fukuda, Gärtner, Lüthi 2008)

In a K-USO:

- *There are no directed cycles.*
- *Every directed path from* $00\ldots0$ *to the sink has length at most* n*.*
- *Every directed path has length at most* $2n$*.*

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Corollary

The simple principal pivoting method with any pivot rule *solves K-LCP in at most* $2n$ *iterations.*

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- There tends to be a “bad example” for any studied deterministic pivot rule.
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Open problems

- Is there a polynomial-time algorithm for P-LCP?
- Is there a deterministic pivot rule with which simple principle pivoting needs a polynomial number of iterations?
- Is there a randomised pivot rule with a polynomial expected number of iterations?
- In particular, is RANDOM PERMUTATION such a rule?