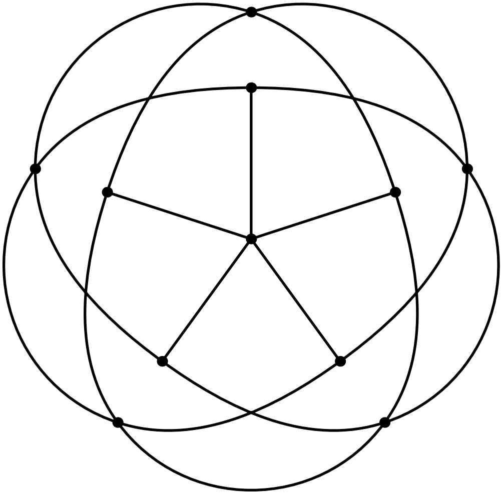
ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

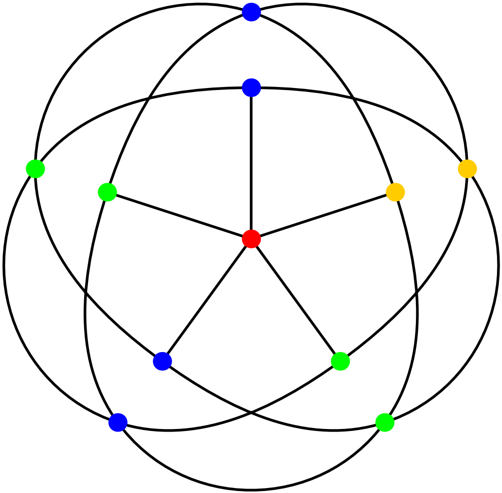
IFOR

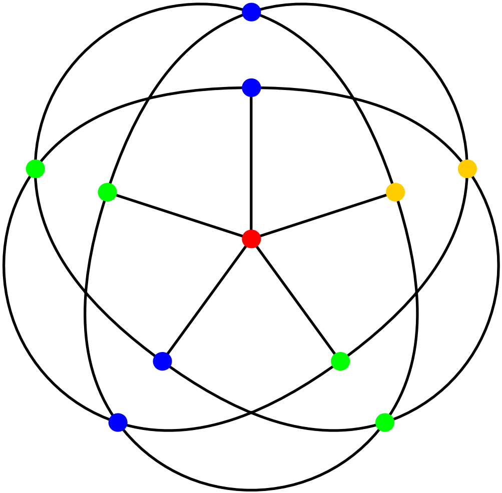
Splitting maximal antichains in the homomorphism order

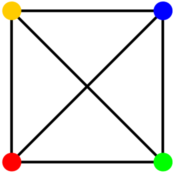
Jan Foniok
joint work with Jaroslav Nešetřil and Claude Tardif

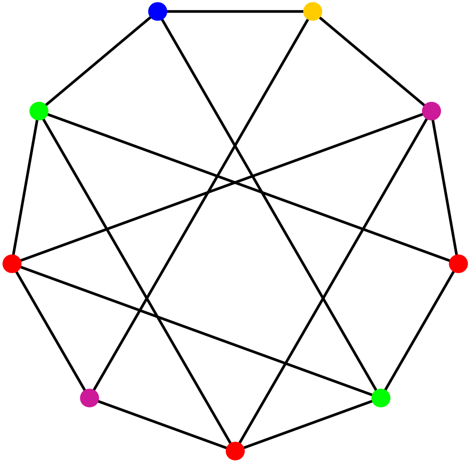
Kolloquium über Kombinatorik 2008, Magdeburg

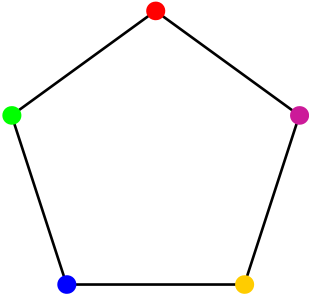












Definition

A homomorphism from G to H is a mapping

$$f : V(G) \to V(H)$$

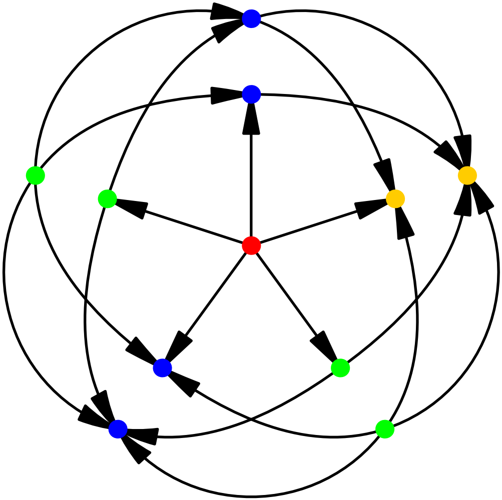
such that

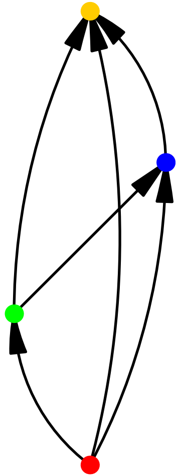
$$uv \in V(G) \implies f(u)f(v) \in V(H).$$

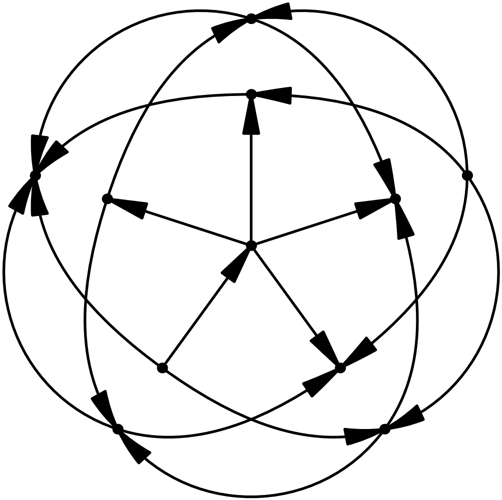
Notation

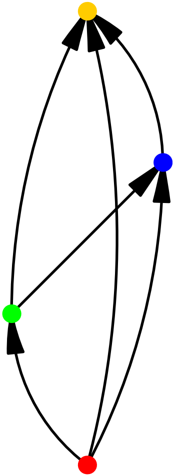
$$f : G \to H$$

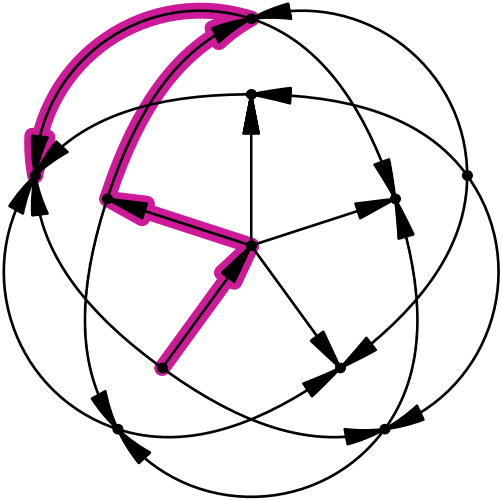
$$G \to H$$

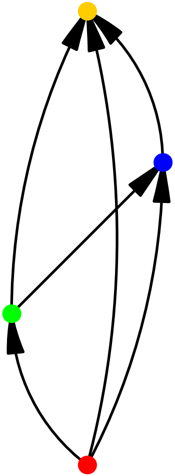












Definition

A duality pair is a pair (F, D) of digraphs, satisfying

$$F \to X \quad \text{if and only if} \quad X \nrightarrow D$$

for any digraph X.

Definition

A duality pair is a pair (F, D) of digraphs, satisfying

$$F \to X \quad \text{if and only if} \quad X \not\to D$$

for any digraph X.

Example (Duality pairs)

$F =$ 

$D =$ 

Definition

A duality pair is a pair (F, D) of digraphs, satisfying

$$F \to X \quad \text{if and only if} \quad X \nrightarrow D$$

for any digraph X.

Example (Duality pairs)

$F =$ 

$D =$ 

Definition

A duality pair is a pair (F, D) of digraphs, satisfying

$$F \to X \quad \text{if and only if} \quad X \not\to D$$

for any digraph X.

Example (Duality pairs)

$F =$ 

$D =$ 

Definition

A duality pair is a pair (F, D) of digraphs, satisfying

$$F \to X \quad \text{if and only if} \quad X \nrightarrow D$$

for any digraph X.

Example (Duality pairs)

$$F = \vec{P}_k \qquad\qquad D = \vec{T}_k$$

(Nešetřil, Pultr, 1978)

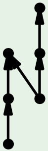
Definition

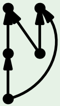
A duality pair is a pair (F, D) of digraphs, satisfying

$$F \to X \quad \text{if and only if} \quad X \nrightarrow D$$

for any digraph X.

Example (Duality pairs)

$F =$ 

$D =$ 

Definition

A duality pair is a pair (F, D) of digraphs, satisfying

$$F \to X \quad \text{if and only if} \quad X \nrightarrow D$$

for any digraph X.

A finite homomorphism duality is a pair $(\mathcal{F}, \mathcal{D})$ of finite sets of digraphs, such that

- no homomorphisms exist between elements of $\mathcal{F}$,
- no homomorphisms exist between elements of $\mathcal{D}$, and
- for any digraph X

$$\mathcal{F} \to X \quad \text{if and only if} \quad X \nrightarrow \mathcal{D}.$$

Definition

A finite homomorphism duality is a pair $(\mathcal{F}, \mathcal{D})$ of finite sets of digraphs, such that

- no homomorphisms exist between elements of $\mathcal{F}$,
- no homomorphisms exist between elements of $\mathcal{D}$, and
- for any digraph X

$$\mathcal{F} \to X \quad \text{if and only if} \quad X \nrightarrow \mathcal{D}.$$

Notation

- $\mathcal{F} \to X$ means: there exists $F \in \mathcal{F}$ such that $F \to X$.
- $X \nrightarrow \mathcal{D}$ means: $X \to D$ for no $D \in \mathcal{D}$

Theorem (Characterisation of finite homomorphism dualities)

1. *(Komárek, 1987)*
   - *If* (F, D) *is a duality pair, then* F *is a tree.*
   - *For any tree* F *there exists unique* D *(the dual of* F*) s.t.* (F, D) *is a duality pair.*

2. *(Nešetřil, Tardif, 2000; JF, Nešetřil, Tardif, 2006)*
   - *If* $(\mathcal{F}, \mathcal{D})$ *is a finite duality, then* $\mathcal{F}$ *is a set of forests and* $\mathcal{D}$ *is uniquely determined by* $\mathcal{F}$*.*
   - *If* $\mathcal{F}$ *is a set of forests, then there exists* $\mathcal{D}$ *s.t.* $(\mathcal{F}, \mathcal{D})$ *is a finite duality.* *(Construction was provided.)*

Homomorphic equivalence

$G \to H$ ... there exists a homomorphism from G to H

$G \sim H$ ... $G \to H$ and $H \to G$

The existence of a homomorphism defines a relation on the class of all digraphs. This relation is reflexive and transitive (it is a preorder).

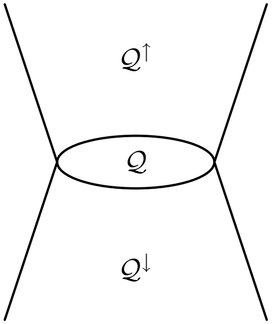
Definition

The homomorphism order is the partial order induced by the existence of a homomorphism on the set of all $\sim$-equivalence classes (or, equivalently, on the set of all cores).

Maximal antichains in the homomorphism order

A finite set $\mathcal{Q}$ of digraphs is a maximal antichain if

1. any two digraphs in $\mathcal{Q}$ are incomparable,
2. any digraph not in $\mathcal{Q}$ is comparable with some digraph in $\mathcal{Q}$.



Maximal antichains in the homomorphism order

A finite set $\mathcal{Q}$ of digraphs is a maximal antichain if

1. any two digraphs in $\mathcal{Q}$ are incomparable,
2. any digraph not in $\mathcal{Q}$ is comparable with some digraph in $\mathcal{Q}$.

Example (Some dualities are maximal antichains)

- $\mathcal{Q} = \{\vec{P}_3, \vec{T}_3\}$
- $\mathcal{Q} = \{\vec{P}_1, \vec{T}_1\}$

Maximal antichains in the homomorphism order

A finite set $\mathcal{Q}$ of digraphs is a maximal antichain if

1. any two digraphs in $\mathcal{Q}$ are incomparable,
2. any digraph not in $\mathcal{Q}$ is comparable with some digraph in $\mathcal{Q}$.

Example (Some dualities are maximal antichains)

- $\mathcal{Q} = \{\vec{P}_3, \vec{T}_3\}$
- $\mathcal{Q} = \{\vec{P}_1, \vec{T}_1\}$ is not an antichain, but $\{\vec{P}_1\}$ is.

Theorem (JF, Nešetřil, Tardif, 2006)

In the homomorphism order of digraphs, all finite maximal antichains $\mathcal{Q}$ *other than* $\mathcal{Q} = \{\vec{P}_0\}$, $\mathcal{Q} = \{\vec{P}_1\}$ *are of the form*

$$\mathcal{Q} = \mathcal{F} \cup \mathcal{D}$$

for some finite homomorphism duality $(\mathcal{F}, \mathcal{D})$.

Theorem (JF, Nešetřil, Tardif, 2006)

In the homomorphism order of digraphs, all finite maximal antichains $\mathcal{Q}$ *other than* $\mathcal{Q} = \{\vec{P}_0\}$, $\mathcal{Q} = \{\vec{P}_1\}$ *are of the form*

$$\mathcal{Q} = \mathcal{F} \cup \mathcal{D}$$

for some finite homomorphism duality $(\mathcal{F}, \mathcal{D})$.

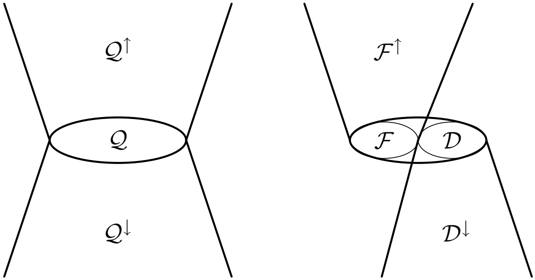
Péter L. Erdős and Lajos Soukup reproved the theorem in 2008 using the sparse incomparability lemma.

Corollary

In the homomorphism order of digraphs, all finite maximal antichains $\mathcal{Q}$ other than $\mathcal{Q} = \{\vec{P}_0\}$, $\mathcal{Q} = \{\vec{P}_1\}$ split.

Corollary

In the homomorphism order of digraphs, all finite maximal antichains $\mathcal{Q}$ other than $\mathcal{Q} = \{\vec{P}_0\}$, $\mathcal{Q} = \{\vec{P}_1\}$ split.



Theorem (JF, Nešetřil, Tardif, 2006)

In the homomorphism order of digraphs, all finite maximal antichains $\mathcal{Q}$ *are of the form*

$$\mathcal{Q} = \mathcal{F} \cup \{D \in \mathcal{D} : D \nrightarrow F \textit{ for any } F \in \mathcal{F}\}$$

for some finite homomorphism duality $(\mathcal{F}, \mathcal{D})$.

Theorem (JF, Nešetřil, Tardif, 2006)

In the homomorphism order of digraphs, all finite maximal antichains $\mathcal{Q}$ *are of the form*

$$\mathcal{Q} = \mathcal{F} \cup \{D \in \mathcal{D} : D \nrightarrow F \text{ for any } F \in \mathcal{F}\}$$

for some finite homomorphism duality $(\mathcal{F}, \mathcal{D})$.

Theorem (JF, Nešetřil, Tardif, 2010)

The same applies to balanced-free antichains in other partial orders, under the following conditions:

- *distributive lattice, and*
- *Heyting operation exists, and*
- *unique connected decompositions exist, and*
- *sparse incomparability axiom applies.*