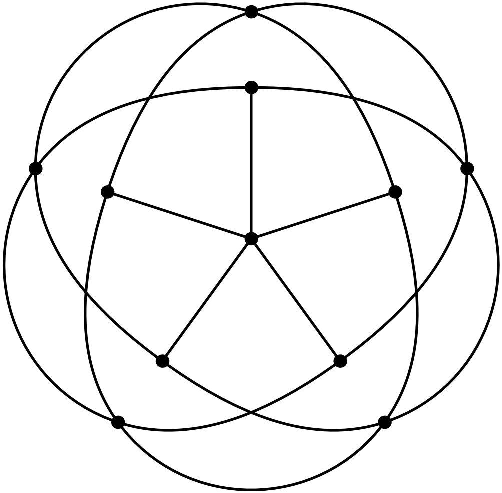
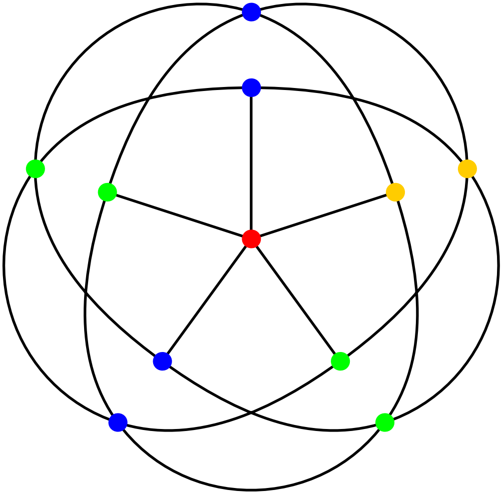
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

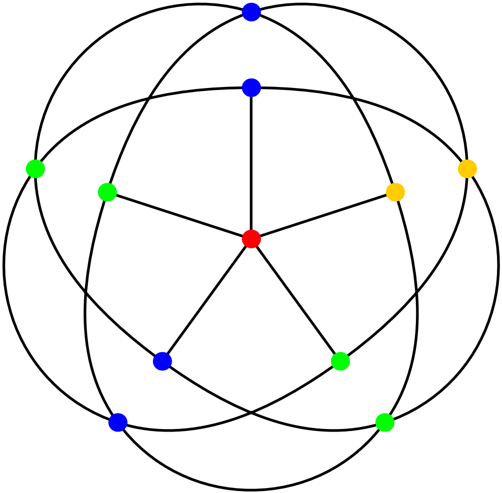
Generating tractable CSPs by means of adjoint functors

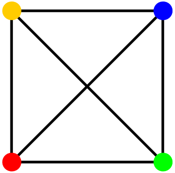
Jan Foniok
joint work with Claude Tardif

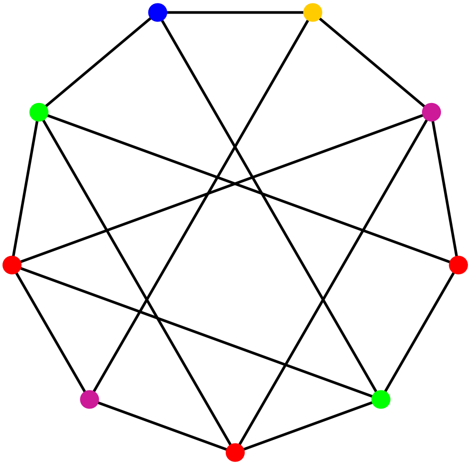
Fête of Combinatorics and Computer Science

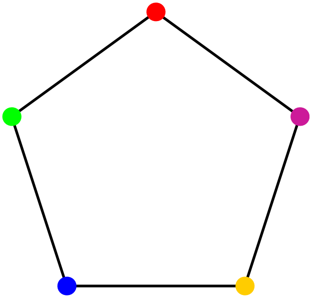












Definition

A homomorphism from G to H is a mapping

$$f : V(G) \to V(H)$$

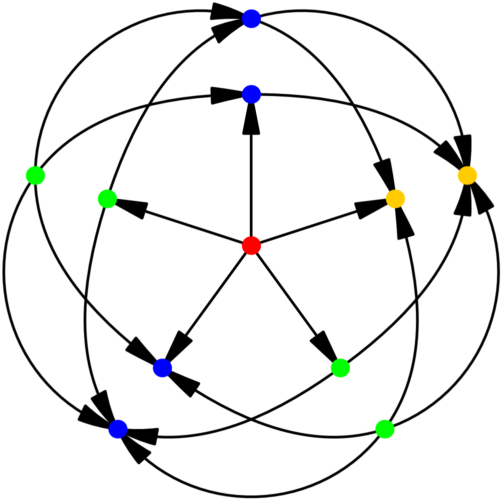
such that

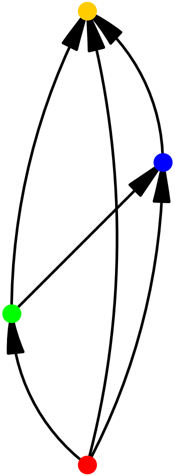
$$uv \in V(G) \quad \Longrightarrow \quad f(u)f(v) \in V(H).$$

Notation

$$f : G \to H$$

$$G \to H$$





Constraint Satisfaction Problem CSP($\mathbb{H}$)

$\mathbb{H}$ is a fixed graph (a template)

Instance: a graph $\mathbb{G}$

Question: $\mathbb{G} \to \mathbb{H}$?

Constraint Satisfaction Problem CSP($\mathbb{H}$)

$\mathbb{H}$ is a fixed graph (a template)

Instance: a graph $\mathbb{G}$

Question: $\mathbb{G} \to \mathbb{H}$?

Theorem (Hell, Nešetřil, 1990)

For undirected graphs:

- *if* $\mathbb{H}$ *is bipartite or contains a loop then CSP($\mathbb{H}$) $\in$ P*
- *otherwise CSP($\mathbb{H}$) is NP-complete*

Conjecture (Feder, Vardi, 1998)

For directed graphs:

- *either CSP($\mathbb{H}$) $\in$ P*
- *or CSP($\mathbb{H}$) is NP-complete*

Conjecture (Feder, Vardi, 1998)

For directed graphs:

- *either CSP(*$\mathbb{H}$*)* $\in$ *P*
- *or CSP(*$\mathbb{H}$*) is NP-complete*

Some polynomial cases

- $\mathbb{H}$ has finite duality
- $\mathbb{H}$ has tree duality (Hell, Nešetřil, Zhu, 1996)
- some conditions involving universal algebra (Jeavons, Bulatov, Krokhin, ...)
- ...

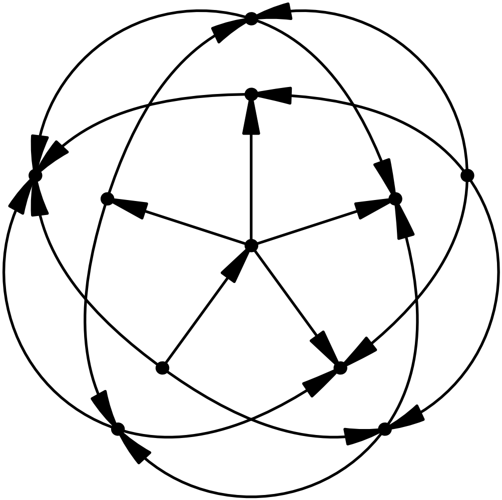
Conjecture (Feder, Vardi, 1998)

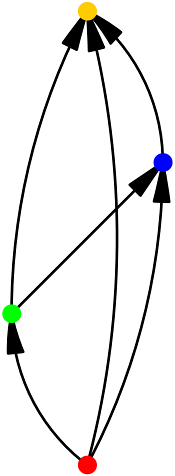
For directed graphs:

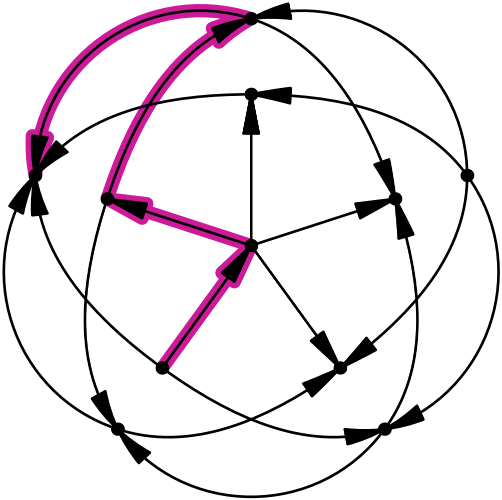
- *either CSP($\mathbb{H}$) $\in$ P*
- *or CSP($\mathbb{H}$) is NP-complete*

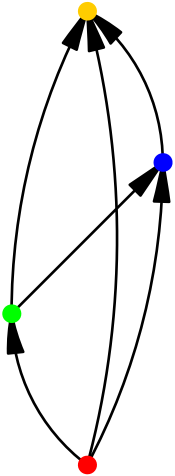
Some polynomial cases

- $\mathbb{H}$ has finite duality
- $\mathbb{H}$ has tree duality (Hell, Nešetřil, Zhu, 1996)
- some conditions involving universal algebra (Jeavons, Bulatov, Krokhin, ...)
- ...









Definition

A complete set of obstructions for a template H is a set $\mathcal{F}$ of digraphs such that

- if $F \in \mathcal{F}$, then $F \nrightarrow H$
- if $G \nrightarrow H$ then there exists $F \in \mathcal{F}$ such that $F \rightarrow G$

Definition

A complete set of obstructions for a template H is a set $\mathfrak{F}$ of digraphs such that

- if $\mathsf{F} \in \mathfrak{F}$, then $\mathsf{F} \nrightarrow \mathsf{H}$
- if $\mathsf{G} \nrightarrow \mathsf{H}$ then there exists $\mathsf{F} \in \mathfrak{F}$ such that $\mathsf{F} \rightarrow \mathsf{G}$

Equivalently

$$\mathsf{G} \nrightarrow \mathsf{H} \iff \exists \mathsf{F} \in \mathfrak{F} : \mathsf{F} \rightarrow \mathsf{G}$$

Definition

A complete set of obstructions for a template H is a set $\mathfrak{F}$ of digraphs such that

- if $F \in \mathfrak{F}$, then $F \nrightarrow H$
- if $G \nrightarrow H$ then there exists $F \in \mathfrak{F}$ such that $F \to G$

Equivalently

$$G \nrightarrow H \iff \exists F \in \mathfrak{F} : F \to G$$

Example: Bipartite graphs

$\mathfrak{F} = \{C_3, C_5, C_7, \ldots\}$ is a complete set of obstructions for $H = K_2$

Definition

A template H has

- finite duality if it has a *finite* complete set $\mathcal{F}$ of obstructions
- tree duality if it has a complete set $\mathcal{F}$ of obstructions that are all *trees*

Definition

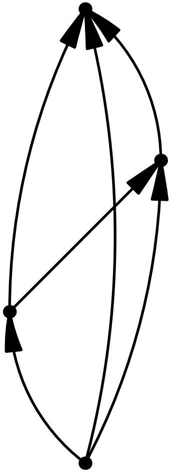
A template H has

- finite duality if it has a *finite* complete set $\mathcal{F}$ of obstructions
- tree duality if it has a complete set $\mathcal{F}$ of obstructions that are all *trees*

Fact

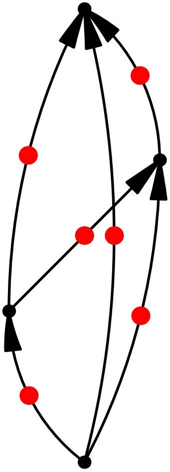
In both cases CSP(H) is polynomial-time solvable

- finite duality: exhaustive search *(check all possible mappings from obstructions)*
- tree duality: consistency check *(Hell, Nešetřil, Zhu, 1996; Feder, Vardi, 1998)*



Arc graph

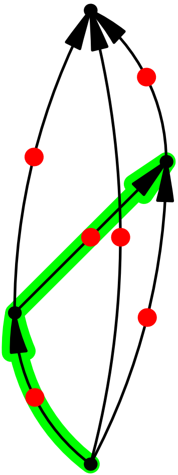
For a digraph H define its arc graph δH:



Arc graph

For a digraph H define its arc graph δH:

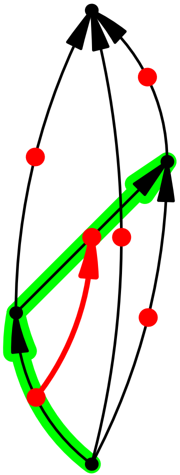
- $V(\delta H) = E(H)$



Arc graph

For a digraph H define its arc graph δH:

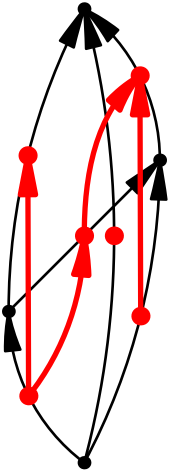
- $V(\delta H) = E(H)$
- e, e' joined by an arc if they form a 2-path in H



Arc graph

For a digraph H define its arc graph δH:

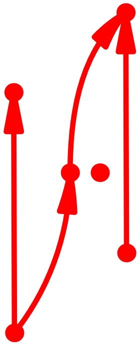
- $V(\delta H) = E(H)$
- e, e' joined by an arc if they form a 2-path in H



Arc graph

For a digraph H define its arc graph δH:

- $V(\delta H) = E(H)$
- e, e' joined by an arc if they form a 2-path in H



Arc graph

For a digraph H define its arc graph δH:

- $V(\delta H) = E(H)$
- e, e' joined by an arc if they form a 2-path in H

Theorem

If a template $\mathbb{H}$ *has tree duality, then its arc graph* $\delta\mathbb{H}$ *also has tree duality.*

Theorem

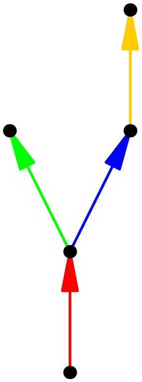
If a template $\mathbb{H}$ *has tree duality, then its arc graph* $\delta\mathbb{H}$ *also has tree duality.*

Proof.

Construct a complete set of tree obstructions for $\delta\mathbb{H}$ from the tree obstructions for $\mathbb{H}$. □

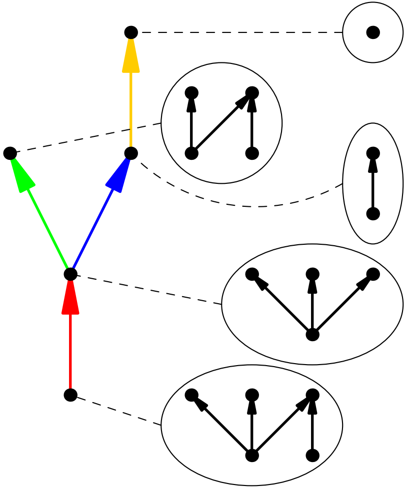
Sproink of a tree

- start with a tree T



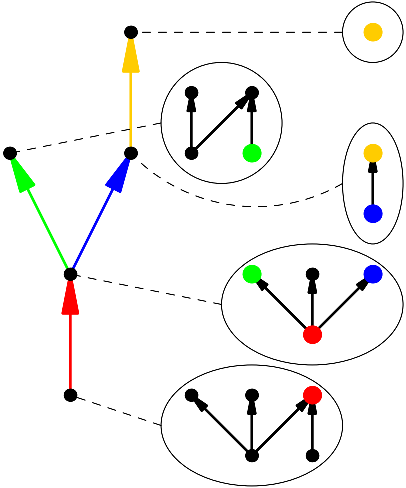
Sproink of a tree

- start with a tree T
- for each vertex u of T, take a tree $F(u)$ of height at most one



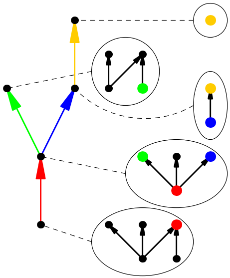
Sproink of a tree

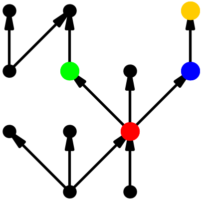
- start with a tree T
- for each vertex u of T, take a tree $F(u)$ of height at most one
- for each edge incident with u, mark a vertex in $F(u)$



Sproink of a tree

- start with a tree T
- for each vertex u of T, take a tree $F(u)$ of height at most one
- for each edge incident with u, mark a vertex in $F(u)$
- identify marked vertices





Sproink of a tree

- start with a tree T
- for each vertex u of T, take a tree $F(u)$ of height at most one
- for each edge incident with u, mark a vertex in $F(u)$
- identify marked vertices

Sproinks of a family

For a family $\mathcal{F}$ of trees, $\mathrm{Sproink}(\mathcal{F})$ is the set of all sproinks of all trees in $\mathcal{F}$.

Theorem

Let H *have tree duality. If* $\mathcal{F}$ *is a complete set of tree obstructions for* H*, then* $\mathrm{Sproink}(\mathcal{F})$ *is a complete set of tree obstructions for* $\delta\mathrm{H}$*, and so* $\delta\mathrm{H}$ *also has tree duality.*

Theorem

Let H *have tree duality. If* $\mathcal{F}$ *is a complete set of tree obstructions for* H*, then* $\mathrm{Sproink}(\mathcal{F})$ *is a complete set of tree obstructions for* δH*, and so* δH *also has tree duality.*

Proof.

Key construction: $G \mapsto \delta^{-1}G$ such that

$$G \to \delta H \quad \text{if and only if} \quad \delta^{-1}G \to H.$$

Then

- if $S \in \mathrm{Sproink}(F)$, then $F \to \delta^{-1}S$; so if $F \nrightarrow H$, then $S \nrightarrow \delta H$,
- if $F \to \delta^{-1}G$, then there is a sproink S such that $S \to G$.

□

Adjoint functors

Use graph operations Ψ for which there exists an "almost inverse" operation Ψ^{-1} such that

$$G \to \Psi H \quad \text{if and only if} \quad \Psi^{-1} G \to H.$$

The precise definition uses category theory notions that we omit here.

Adjoint functors

Use graph operations Ψ for which there exists an "almost inverse" operation Ψ^{-1} such that

$$G \to \Psi H \quad \text{if and only if} \quad \Psi^{-1} G \to H.$$

The precise definition uses category theory notions that we omit here.

Terminology

In the above situation, Ψ is a right adjoint and Ψ^{-1} its left adjoint.

Right adjoints: Construction (Aleš Pultr, 1970)

The operation Ψ is defined by:

- two graphs P, Q,
- two homomorphisms $\phi_1, \phi_2 : P \to Q$.

Then for a graph H we define:

- $V(\Psi H) = \{f : f \text{ is a homomorphism } P \to H\}$,
- (f_1, f_2) is an arc if and only if there exists $g : Q \to H$ such that $f_1 = g \circ \phi_1$ and $f_2 = g \circ \phi_2$.

Left adjoints

Let $\Psi^{-1}G$ contain a copy P_v of P for each vertex v of G, and a copy Q_a of Q for each arc a of G.

For each arc $a = (u, v)$ identify vertices in P_u and P_v with the corresponding vertices in Q_a.

Left adjoints

Let $\Psi^{-1}G$ contain a copy P_v of P for each vertex v of G, and a copy Q_a of Q for each arc a of G.

For each arc $a = (u, v)$ identify vertices in P_u and P_v with the corresponding vertices in Q_a.

Lemma

Then

$$G \to \Psi H \quad \textit{if and only if} \quad \Psi^{-1}G \to H. \tag{$*$}$$

Lemma

Then

$$\mathbb{G} \to \Psi\mathbb{H} \quad \textit{if and only if} \quad \Psi^{-1}\mathbb{G} \to \mathbb{H}. \tag{$*$}$$

Theorem

If CSP($\mathbb{H}$) is solvable in polynomial time, then CSP($\Psi\mathbb{H}$) is also solvable in polynomial time.

Lemma

Then

$$\mathbb{G} \to \Psi\mathbb{H} \quad \textit{if and only if} \quad \Psi^{-1}\mathbb{G} \to \mathbb{H}. \qquad (*)$$

Theorem

If CSP($\mathbb{H}$) is solvable in polynomial time, then CSP($\Psi\mathbb{H}$) is also solvable in polynomial time.

Proof.

Observe that $\Psi^{-1}\mathbb{G}$ is polynomial in the size of $\mathbb{G}$ and can be constructed in polynomial time. Use $(*)$. □

Theorem

If the template $\mathbb{H}$ *has tree duality, then* $\Psi\mathbb{H}$ *also has tree duality.*

Theorem

If the template H *has tree duality, then* $\Psi\mathrm{H}$ *also has tree duality.*

Proof.

Use a characterisation by Feder, Vardi:

$$\mathrm{H} \text{ has tree duality} \iff \mathfrak{U}\mathrm{H} \to \mathrm{H}$$

Check that if $\mathfrak{U}\mathrm{H} \to \mathrm{H}$ then $\mathfrak{U}\Psi\mathrm{H} \to \Psi\mathrm{H}$. □

Summary

- Adjoint functors are special graph operations (defined in categories).
- Right adjoints preserve polynomial-time templates as well as tree duality.
- The arc graph is a special case of such a right adjoint.
- In the case of the arc graph we have an explicit construction of the tree obstructions.

Summary

- Adjoint functors are special graph operations (defined in categories).
- Right adjoints preserve polynomial-time templates as well as tree duality.
- The arc graph is a special case of such a right adjoint.
- In the case of the arc graph we have an explicit construction of the tree obstructions.

Open question

Is there an explicit description of obstructions for general adjoint functors?