Structural aspects of the class of relational structures

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Abstract. We summarise some older results and bring a new result concerning the characterisation of small maximal antichains in the homomorphism order of Δ-structures for various types Δ.

1. Introduction

Let t be a positive integer. A Δ-*structure*, also called a *relational structure* of type $\Delta = (\delta_1, \delta_2, \dots, \delta_t)$, is a pair $A = (V, (R_1, R_2, \dots, R_t))$ where V is a finite set and $R_i \subseteq V^{\delta_i}$ for $i = 1, 2, \dots, t$. The concept of Δ-structure is a generalisation of directed graphs. Δ-structures are directed hypergraphs with edges of t types (colours). The number of types of edges t is denoted by $|\Delta|$. As Δ-structures resemble graphs, we often call the elements of V *vertices* and the elements of R_i's *edges*. Notice that a directed graph is a (2)-structure. The symbols $V(A)$ for the vertex set of A and $R_i(A)$ for the i-th edge set of A will be used.

Let A and B be Δ-structures of the same type. A mapping $f : V(A) \to V(B)$ is a *homomorphism* of A to B if for every $1 \leq i \leq t$ it satisfies the condition that if $(v_1, v_2, \dots, v_{\delta_i}) \in R_i(A)$ then $(f(v_1), f(v_2), \dots, f(v_{\delta_i})) \in R_i(B)$. In other words, a homomorphism is a mapping that preserves edges. The fact that f is a homomorphism of A to B is denoted by $f : A \to B$. The existence of any such homomorphism is denoted by $A \to B$ or $A \leq B$; we say that A *is homomorphic to* B. If $A \to B$ and $B \to A$, then we say that A and B are *homomorphically equivalent* and write $A \sim B$. A bijective homomorphism whose inverse is a homomorphism as well is called an *isomorphism*; if there exists an isomorphism of A to B, we say that A and B are *isomorphic* and write $A \cong B$.

As the identity mapping is a homomorphism of A to A and the composition of two homomorphisms is a homomorphism as well, the relation $\leq$ is a quasiorder on the class of all Δ-structures of a given type Δ.

Let A and B be two Δ-structures. We say that B is a *substructure* of A if $V(B) \subseteq V(A)$ and $R_i(B) \subseteq R_i(A)$ for every i. A substructure B of A is an *induced substructure* of A if $R_i(B) = R_i(A) \cap V(B)^{\delta_i}$.

A Δ-structure C is a *core* if it is not homomorphic to any of its proper subgraphs.

The following three facts are well known and straightforward: A Δ-structure C is a core if and only if every *endomorphism* of C (a homomorphism of C to C) is an *automorphism* (an isomorphism of C to C). Two cores are homomorphically equivalent if and only if they are isomorphic. Every Δ-structure A is homomorphically equivalent to a unique (up to isomorphism) core C; C is an induced substructure of A and is called *the core of* A.

Because of the second of these facts, the relation $\leq$ is a partial order on the set of (isomorphism classes of) all cores of a given type Δ. In this paper, we study some properties of this partial order, called the *homomorphism order*.

2. Homomorphism duality

To describe some of the properties of the homomorphism order, we will make use of the concept of homomorphism duality. This concept was introduced by *Nešetřil and Pultr* [1978].

We say that a pair (P, D) of Δ-structures is a *duality pair* if for an arbitrary Δ-structure A it is true that $P \to A$ if and only if $A \nrightarrow D$.

All duality pairs were described by *Nešetřil and Tardif* [2000] in the following way.

The *shadow* of a Δ-structure A is the undirected multigraph $G(A)$ whose vertices are the elements of $V(A)$ and there is an edge between a and b for every δ_i-tuple $(a_1, \dots, a_n) \in R_i$ and for every j such that $a_j = a$, $a_{j+1} = b$, $\delta_i \geq 2$ and R_i is an edge set of A. The Δ-structure A is called a Δ-*tree* if its shadow $G(A)$ is a tree.

Example. Let $\Delta = (1, 2, 3)$, let A be a Δ-structure,

$$A = (\{1, 2, \dots, 6\}, (\{(1), (3), (5)\}, \{(3, 2), (3, 6), (6, 5)\}, \{(1, 5, 6), (4, 4, 1), (4, 5, 2)\})).$$

The shadow $G(A)$ of the $(1, 2, 3)$-structure A is shown in Figure 1. The loop at the vertex 4 is caused by the triple $(4, 4, 1) \in R_3(A)$. Notice that 1-ary relations have no effect on the shadow of the Δ-structure.

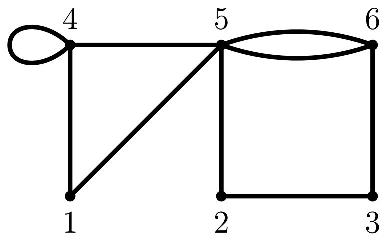

Figure 1. An example of shadow

Let A and B be Δ-structures. The *B-th power* A^B of A has the property that for any Δ-structure C, we have $B \times C \to A$ if and only if $C \to A^B$. The base set $V(A^B)$ is the set of all functions from $V(B)$ to $V(A)$; for $i \leq t$ we have $(f_1, \dots, f_{\delta_i}) \in R_i(A^B)$ if and only if whenever $(b_1, \dots, b_{\delta_i}) \in R_i(B)$, then also $(f_1(b_1), \dots, f_{\delta_i}(b_{\delta_i})) \in R_i(A)$.

Theorem 1 (Nešetřil and Tardif [2000]) *Let A be a Δ-structure. There exists a Δ-structure B such that (A, B) is a duality pair if and only if A is homomorphically equivalent to a Δ-tree. Then B is unique and it is homomorphically equivalent to C^A for some Δ-structure C satisfying the condition that whenever $C \to D \to A$ then either $D \to C$ or $A \to D$.*

Homomorphism dualities' connection to complexity issues has attracted an interest of its own. We shall use the dualities for another purpose.

3. Maximal antichains of small size

Let $(P, \leq)$ be a partially ordered set. Two elements $x, y \in P$ are *incomparable* if neither $x \leq y$ nor $y \leq x$. This fact is denoted by $x \parallel y$. A subset $S \subseteq P$ is an *antichain* or an *independent set* if any two elements $s, t \in S$ are incomparable. An antichain $S \subseteq P$ is *maximal* if $S \cup \{x\}$ is not an antichain for any $x \in P \setminus S$.

Let $(P, \leq)$ be a finite partially ordered set. Define a binary relation $\lhd$ on the set P: $x \lhd y$ if and only if $x \leq y$ and there is no $z \in P$ such that $x \leq z \leq y$. We have $x \leq y$ if and only if there is a directed path from x to y in the directed graph $(P, \lhd)$. Maximal elements in $(P, \leq)$ are vertices with outdegree 0 in $(P, \lhd)$ and minimal elements are vertices with indegree 0. A maximal antichain in $(P, \leq)$ is a minimal cut separating all minimal elements from all maximal elements in the directed graph $(P, \lhd)$.

We are interested in maximal antichains of a given (small) size in the homomorphism order of Δ-structures. The most important result about this topic, attained by *Nešetřil and Tardif* [2001], is the following theorem. Note that P_k denotes the directed path of length k.

Theorem 2 (Nešetřil and Tardif [2001]) *The maximal antichains of size two in the homomorphism order of directed graphs are the pairs $\{T, D_T\}$ where T is a core tree different from P_0, P_1, P_2 and (T, D_T) is a duality pair.*

Notice that D_T is uniquely determined by T (Theorem 1). It is obvious that if $\{T, D_T\}$ is an antichain then it is maximal, as by the definition of duality for any other digraph G either $T \leq G$ or $G \leq D_T$. It is easy to show that $T \parallel D_T$ unless $T \in \{P_0, P_1, P_2\}$. Therefore the main result is that there are no other maximal antichains of size two.

A somewhat generalised concept of t-graphs was introduced by *Foniok* [2003]. A *t-graph* is a relational structure of type $(2, 2, \ldots, 2)$ (a t-tuple of 2's).

An analogue of Theorem 2 is not true for t-graphs. There are other maximal 2-antichains than the duality pairs.

Example. Let A_1 be a 2-graph consisting of a single edge of type 1 and A_2 a single edge of type 2; i.e.

$$\begin{aligned} A_1 &= (\{u,v\}, (\{(u,v)\}, \emptyset)), \\ A_2 &= (\{u,v\}, (\emptyset, \{(u,v)\})). \end{aligned}$$

Then $A_1 \parallel A_2$ but they do not form a duality pair. It remains open to describe all such exceptions.

Let us now concentrate on maximal antichains of size 1. For structures of type $\Delta = (k)$, where $k \geq 2$, we can use the following lemmas.

Lemma 1 *Let A and B be Δ-structures. If $G(A) \parallel G(B)$, then $A \parallel B$.*

Proof. A homomorphism $f : A \to B$ is also a homomorphism of $G(A)$ to $G(B)$; therefore $A \leq B$ implies $G(A) \leq G(B)$ and $B \leq A$ implies $G(B) \leq G(A)$. $\square$

Lemma 2 *Let G be a non-bipartite undirected graph without loops. Then there exists a graph H incomparable with G, i.e. such that $G \nrightarrow H \nrightarrow G$.*

Proof. By a famous theorem of *Erdős* [1959], there exists a graph H with odd girth larger than the odd girth of G and with chromatic number larger than the chromatic number of G. Then $G \nrightarrow H$ because G contains an odd cycle and the homomorphic image of an odd cycle contains an odd cycle of the same or smaller length. However, H does not contain a short odd cycle.

Let the chromatic number of G be $\chi(G) = k$. Then $G \to K_k$ and $H \nrightarrow K_k$ and therefore $H \nrightarrow G$. $\square$

For directed graphs, there are only finitely many maximal antichains of size 1.

Proposition 1 (Nešetřil and Zhu [1996]) *The only maximal antichains of size 1 in the homomorphism order of digraphs are directed paths of length 0, 1, and 2 (P_0, P_1 and P_2) and a single vertex with a loop (L).*

Proof. Obviously P_0 (the least element), P_1, P_2 and L (the greatest element) form one-element maximal antichains.

Let A be a core digraph different from any of these four graphs. Then A does not contain a loop as the only core with a loop is L. Let $G = G(A)$ be the shadow of A. If G is non-bipartite, Lemmas 1 and 2 apply. Any orientation B of a graph H such that $G \parallel H$, is incomparable with A. We need not care about multiple edges in the shadow of A; we can safely consider them as single edges.

If G is bipartite and A contains an unbalanced cycle, let H be a graph with $\chi(H) > \chi(G)$ and such that the girth of H is greater than the length of the shortest unbalanced cycle in A. Let B be an arbitrary orientation of H. Then $A \nrightarrow B$ as the homomorphic image of an unbalanced cycle contains an unbalanced cycle of the same or smaller length. If $B \to A$, then $H \to G$ and $\chi(H) \le \chi(G)$ and that is a contradiction.

Finally, let A be balanced. For every balanced digraph B there exists a unique mapping l_B of the vertex set $V(B)$ to the set of non-negative integers $\mathbb{Z}^*$ such that $l_B(v) = l_B(u) + 1$ whenever (u, v) is an edge of B and there exists a vertex x for which $l_B(x) = 0$. Such a mapping is called the *level mapping* of B. The *height* of B is $ht(B) = \max\{l(u) : u \in V(B)\}$.

Homomorphisms of digraphs preserve level distance, i.e. if $f : A \to B$ is a homomorphism, then $l_B(f(u)) - l_B(f(v)) = l_A(u) - l_A(v)$. Therefore if $A \to B$, then $ht(A) \le ht(B)$.

An *oriented path* is an orientation of a path. Let P be an oriented path. We pick one of the two end vertices and call it the *initial vertex* of P. The other end vertex will be called the *terminal vertex* of P. Now every edge of P is either a *forward edge* or a *backward edge*, depending on whether it heads in the direction from the initial vertex to the terminal vertex or vice versa. Each oriented path will be assigned a code which is a sequence of zeroes and ones: traversing form the initial vertex to the terminal vertex, zero means a forward edge and one a backward edge. Of course, paths with codes $(a_1, a_2, \dots, a_k)$ and $(1 - a_k, 1 - a_{k-1}, \dots, 1 - a_1)$ are isomorphic with the initial and the terminal vertex swapped.

Oriented paths are usually displayed with all edges drawn upwards, as in Figure 2. This path has two vertices in level 0, four in level 1, two in level 2 and one in level 3.

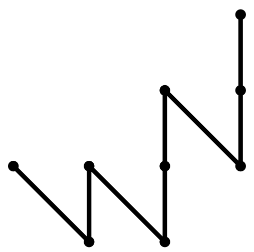

Figure 2. An example of an oriented path (with the code 10100100)

We will now finish the proof by providing a construction of an incomparable digraph for a balanced digraph A. Let d be the minimal number of edges joining a vertex in level k with a vertex in level $k + 3$ in A, $k = 0, 1, \dots, ht(A) - 3$, and let $c = 00101 \dots 0101$ ($(d + 1)/2$ zeroes and $(d - 1)/2$ ones). Let P be the path with the code

$$\underbrace{ccc \dots cc}_{ht(A)\text{ times}} 00.$$

Then $ht(P) > ht(A)$, so $P \nrightarrow A$. The minimum subpath joining a vertex in level k with a vertex in level $k+3$ is called a *zigzag*. A zigzag is mapped by a homomorphism to a zigzag of the same or smaller length; however, the shortest zigzag in P is longer than the shortest zigzag in A and so $A \nrightarrow P$. □

If A is a Δ-structure with $|\Delta| = 1$, $\Delta = (k)$, $k \ge 3$, as a consequence of Lemma 1, we get that if $G(A)$ does not contain a loop and is not bipartite, then $\{A\}$ is not a maximal antichain.

There are only trivial maximal antichains of size 1 for Δ-structures with $|\Delta| > 1$. Note that a *loop* is an edge in the form $(x, x, \dots, x)$ for a vertex x.

Proposition 2 *Let $t \ge 2$, $\Delta = (\delta_1, \delta_2, \dots, \delta_t)$ and $\delta_1 \ge 2$. Then the only two maximal antichains of size one in the homomorphism order of Δ-structures are a vertex with no edges and a vertex with all loops.*

Proof. Let $A = (V, (R_1, R_2, \dots, R_t))$ be a Δ-structure that is a core. We need to show that unless $A = E = (\{v\}, (\emptyset, \emptyset, \dots, \emptyset))$ or $A = K_1(\Delta) = (\{v\}, (V^{\delta_1}, V^{\delta_2}, \dots, V^{\delta_t}))$, there is a Δ-structure B such that $A \parallel B$.

First, if there exists $1 \leq i \leq t$ such that $R_i = \emptyset$, then the structure B with $V(B) = \{u\}$, $R_i(B) = \{u\}^{\delta_i}$ and $R_j(B) = \emptyset$ for $j \neq i$ is incomparable with A. The same structure is incomparable with A if A has edges of all types but there is no loop $(a, a, \ldots, a) \in R_i$.

Now suppose that A has loops of all types (colours). As $A \nsim K_1(\Delta)$, there is no vertex in $V(A)$ with all loops. Let k be the number of vertices that have loops in colours $2, 3, \ldots, t$, i.e. $k = |K|$ where

$$K = \{u \in V : (\forall i \in \{2, 3, \ldots, t\})\ (u, u, \ldots, u) \in R_i\}.$$

Let

$$\begin{aligned} V(B) &= \{0, 1, 2, \ldots, k\}, \\ R_1(B) &= \{(a, b, b, \ldots, b) : 0 \leq a < b \leq k\}, \\ R_i(B) &= V(B)^{\delta_i} \text{ for } 2 \leq i \leq t. \end{aligned}$$

We know that $A \nrightarrow B$ because there is no loop in $R_1(B)$. If there existed a homomorphism $f : B \to A$, it would have to map all vertices of B to the subset $K \subseteq V(A)$. The mapping f cannot be injective because $|V(B)| > |K|$. Therefore there exist $u, v \in V(B)$ such that $u < v$ and $f(u) = f(v) = x \in K$. As $(u, v, v, \ldots, v) \in R_1(B)$ and f is a homomorphism, $(f(u), f(v), f(v), \ldots, f(v)) = (x, x, \ldots, x) \in R_1(A)$ and the vertex x has all loops, contradicting the fact that A was not homomorphically equivalent to $K_1(\Delta)$. $\square$

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