About maximal cliques in a certain class of circular-arc graphs

Gabrio Caimi, Martin Fuchsberger, Rico Zenklusen
Institut für Operations Research, ETH Zurich,
CH-8092 Zürich

December 8, 2008

Abstract

In this paper we address the problem of finding all maximal cliques in a subclass of circular-arc graphs. We consider only circular-arc graphs that are generated with an arc model where no three arcs cover the whole circle. We prove that the number of maximal cliques is bounded by the number of vertices and that for each maximal clique it exists a point in the circle which is covered by all arcs corresponding to vertices of the given clique.

1 Introduction

The *intersection graph* of a finite family $\mathcal{S}$ of sets is the graph G obtained by introducing for every set $S \in \mathcal{S}$ a corresponding vertex, and two vertices of G are connected if their corresponding sets overlap. A circular-arc graph is the intersection graph of a set of arcs on the circle. Since the size of the circle does not matter, we can always consider unit circles. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of arcs on the circle. We denote by $G(A)$ the circular-arc graph that corresponds to A. We say that some set of arcs A on the circle is an arc model for some given graph G, if $G(A)$ and G are isomorph. An arc model is called proper, if no arc in the arc model contains another arc of the given arc model. Furthermore, we call an arc model *non-degenerate*, if no three arcs of the arc model cover the whole circle.

Circular-arc graphs are a natural generalization of interval graphs. If a circular-arc graph G has an arc model that leaves some point of the circle uncovered, the circle can be cut at that point, which results in an interval representation. Unlike interval graphs, however, circular-arc graphs are not always perfect graphs since even all odd chordless cycles are circular-arc graphs. In general, the number of maximal cliques in a circular-arc graph can grow exponentially in the size of the graph [2]. In this paper, we consider circular-arc graphs that admit a non-degenerate arc model. We call this proper subclass of the set of circular arc graphs *non-degenerate* circular-arc graphs. The set of all non-degenerate circular-arc graphs will be denoted by $\mathcal{C}_{nd}$. Since $\mathcal{C}_{nd}$ still contains all chordless odd cycles, non-degenerate circular-arc graphs are in general not perfect. However, graphs in $\mathcal{C}_{nd}$ share many interesting properties with interval graphs. In particular we show that for any non-degenerate circular-arc, the number of maximal cliques is at most the number of vertices in the graph. Additionally, we show that for any non-degenerate arc model A and any maximal clique in $G(A)$, it exists a point in the circle that is covered by all arcs in A which corresponds to the vertices in the given clique. As a consequence, it is easy to find all maximal cliques of a given non-degenerate circular-arc graph if a corresponding non-degenerate arc model is given.

Throughout this paper we assume that in all arc model we consider, no two arcs have a common endpoint. Any given arc model can easily be modified, witout changing the topology of the represented graph, to satisfy this conditions by slightly moving the endpoints. Let A be an arc model. For some given point p on the circle we call the *overlap set of A corresponding to p* (or simply *overlap set corresponding to p* if there is no danger of ambiguity) the subsets of arcs in A that contain p. The overlap set of A corresponding to p will be denoted by $A(p)$ and p is called a *covering point* of $A(p)$. For a given arc $a \in A$, we call its clockwise endpoint the *left endpoint* and the counterclockwise endpoint the *right endpoint*. We denote the left and right endpoint of a by $l(a)$ and $r(a)$. For two points p_1, p_2 on the circle we denote by $[p_1, p_2]$ the circular arc consisting of the points on the circle that are encountered when going in counterclockwise sense from p_1 to p_2. Analogously, we denote by (p_1, p_2) the set $[p_1, p_2] \backslash \{p_1, p_2\}$.

2 Proof

Theorem 1 (Existence of a covering point) *Let A be a non-degenerate arc model. For every maximal clique Q in $G(A)$, there exists a point p on the circle that is contained in all arcs of A that correspond to vertices in Q.*

Proof: We will show that no clique C in $G(A)$ corresponds to a set of arcs $A_C \subseteq A$ that cover the whole circle. The result then easily follows by observing that $G(A_C)$ corresponds to an interval graph and that all maximal cliques in interval graphs correspond to overlapping sets [1]. By contradiction we assume that there exists a set of pairwise overlapping arcs A' in A that cover the whole circle. Without loss of generality we choose A' to be a minimal set of arcs with this property. Since A is a non-degenerate arc model we must have $|A'| \geq 4$. By minimality of A' we have that every arc $a' \in A'$ contains a point $p_{a'}$ which is not contained in any other arc of A'. Consider an arbitrary set of four arcs $\{a'_1, a'_2, a'_3, a'_4\} \subseteq A'$ where these arcs are numbered in such a way that the four points $p_{a'_1}, p_{a'_2}, p_{a'_3}, p_{a'_4}$ are encountered in this order when going around the circle in counterclockwise sense beginning at $p_{a'_1}$. By definition of the points $p_{a'_i}$, we have that $a'_2 \subset (p_{a'_1}, p_{a'_3})$ and $a'_4 \subset (p_{a'_3}, p_{a'_1})$. This implies that a'_2 and a'_4 do not overlap, thus leading to a contradiction. $\square$

Theorem 2 (Number of maximal cliques in $\mathcal{C}_{nd}$) *Let $G \in \mathcal{C}_{nd}$. The number of maximal cliques of G is bounded by the size of G.*

Proof: Let A be a non-degenerated arc model with corresponding to G. By Theorem 1 maximal cliques in G correspond to maximal overlap sets in A. The theorem will be proven by showing that every maximal overlap set is of the form $A(u)$, where u is the left endpoint of some arc in A. Let p be some point on the circle such that $A(p)$ corresponds to a maximal overlap set and let u be the first point in $\cup_{a\in A}\{l(a), r(a)\}$ that is encountered when going from p along the circle in clockwise sense. We clearly have $A(p) = A(u)$. The point u cannot be a right endpoint of one of the intervals of A since in this case, by choosing a point p' on the circle lying slightly to the left of u, we have $|A(p')| = |A(p)| + 1$, contradicting the maximality of $A(p)$. Thus, u must be a left endpoint of one of the intervals in A. $\square$

Notice that the proof of Theorem 2 suggests a simple algorithm for finding all maximal cliques of a non-degenerate circular-arc graph that is given by a non-degenerate arc-model A: One just has to consider the overlap sets corresponding to all left (or right) endpoints of the arcs $a \in A$ and take the maximal ones.

References

[1] Alexander Schrijver. *Combinatorial Optimization.* Springer, 2003.

[2] A. Tucker. An efficient test for circular-arc graphs. *SIAM J. Computing*, 9(1):1–24, 1980.