Pricing electricity risk by interest rate methods

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Abstract

We address a method for pricing electricity contracts based on valuation of *ability to produce power*, which is considered as the true underlying for electricity derivatives. This approach shows that an evaluation of free production capacity provides a framework where a change–of–numeraire transformation converts electricity forward market into the common settings of money market modeling. Using the toolkit of interest rate theory, we derive explicit option pricing formulas.

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1 Introduction

Beginning in the nineties a number of countries have deregulated their markets for electrical power. This involved the creation of competitive power markets, where electrical energy is traded as a commodity. One of the most popular financial products for electricity risk management is the *power forward*. The buyer of such an instrument is guaranteed the delivery of a pre–determined amount of electrical energy as a constant flow over a future period of time specified in the contract. The delivery is either physical, or settled financially. The importance of power forwards is comparable to those of forward contracts in other commodity markets since both the buyer and the writer insure themselves against possible harmful future price movements. Similarly, derivative instruments written on power forwards are also widely used to hedge the electricity price risk. However, the valuation of these contracts is still under discussion due to the lack of convincing economical pricing concepts. The point here is that the electrical energy is *not economically storable.* Thus, power forwards with non-overlapping delivery intervals seem to have different underlying commodities (electrical energy, delivered in different periods) without any opportunity to transfer one commodity into the other which makes hedging by commodity storage impossible.

At the first glance, the pricing methodology in the style of interest rate theory seems appropriate, since a forward contract supplying 1 MWh of electrical power within a delivery interval immediately after τ is analogous to a zero–bond maturing at the time τ. However, for power forwards, we observe a peculiar price behavior (see [19]): considering prices as a function of time to maturity, one notices that they fluctuate even near maturity (which is also the delivery start). Clearly, this behavior is impossible for bonds, whose prices converge to one when approaching their maturity date. As a response to this, a line of research (see [6], [21], [3], [17]) has focused on modeling the evolution of the whole forward curve dynamics by

$$\frac{dp_t(\tau)}{p_t(\tau)} = \sum_{i=1}^{d} \sigma_t^i(\tau) dW_t^i, \qquad d \in \mathbb{N}. \tag{1}$$

Here $p_t(\tau)$ denotes the price at time t for the future delivery of 1 MWh of at time $\tau \geq t$, (W_t^i) for $i = 1, \ldots, d$ are Brownian motions under risk–neutral probability measure and $(\sigma_t^i(\tau))$ denotes the volatility term structure. The limit $\lim_{t \uparrow \tau} \sum_{i=1}^{d} \sigma_t^i(\tau)^2$ is positive or even $+\infty$, reflecting the fluctuation of power forward prices near maturity. Specifying volatility structure, the dynamics (1) yields explicit option formulas (see [3]). However, purposing (1) directly, qualitative features of electricity risk remain unconsidered. Among them are questions of existence and interpretation of risk–neutral measures, risk hedging by real assets, explicit connection to interest rate models and reasons for increasing volatility at maturity. Furthermore, the question of valuation and hedging for electricity options is frequently considered as a part of hedging problems in incomplete markets and so electricity risk management is sometimes linked to pricing of weather derivatives and insurance–like instruments. Obviously, this point of view neglects the electricity production process: although electrical energy can not be stored, a hedging is still possible by production capacity investments. That is, a transparent and liquid market for contracts on availability of free electricity production capacities will help to price correctly, to reduce, and to avoid risk resulting from electricity production and trading.

In this work, we respond to these aspects considering pricing of electricity contracts within a production capacity market. It turns out that equilibrium asset prices are given by their future payoff, expected with respect to some equivalent measure. Using this framework, we apply a change–of–numeraire transformation (see [10]) to avail the toolkit of interest rate theory.

Let us mention some relations to other research in this field. In [9], the authors expose questions of electricity pricing and explain that the non–storability requires a modeling of production process. They suggest to use marginal fuel (gas, oil, etc.) prices to describe forward power prices, considering the fact that fuel is easily transformed into electrical power, provided an electricity production unit is rented for the delivery period. The work [22] and the recent paper [7] present a counterpart of this conception for hydro–electric power generation giving a treatment for operational flexibility valuation and dispatch management to hedge electricity contracts by optimal schedule of hydro–electric power plants. Both approaches show that a detailed production–based modeling may shed light on how to price electricity contracts. We follow this insight in our paper. Another line of research (see [21], [2], [15], [5]) focuses on modeling the stochastic process of spot price. Again, in [15] a production–based point of view exhibits reasons for high spot price volatility and highlights the role of real–time electricity auctions. Electricity auctions themselves are considered as a crucial point in deregulation of electricity industry and enjoy a considerable research interest (see [8], [1], and [11] – [14]).

The paper is organized as follows. First we introduce the equilibrium in a production capacity market to derive a principle for contract pricing. In Section 3 we discuss an application of interest rate techniques to valuation of contracts written on electricity. An empirical study of our pricing methodology is presented in Section 4. The detailed proof of the existence of a symmetric equilibrium, on which the contract valuation is based, is given in Section 5. The last Section 6 summarizes our results.

2 An equilibrium principle for pricing electricity contracts

Here we present a valuation principle which comes from the realization that though electricity can not be stored, it can be produced and so the true underlying of electricity contracts will be *physical ability to produce power*. Consequently, writers of contingent claims prefer contracts perfectly replicated by an appropriate portfolio of real production units. For this reason, we observe that in electricity markets, financial assets typically mimic agreements on power production capacities. The market price for such an agreement is to be considered as a fair price for the financial asset which assures the agreements payoff.

We describe electricity market at discrete equidistant times $t = 0, \ldots, T$, $T \in \mathbb{N}$ on the filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t=0}^T)$, where $\mathcal{F}_t$ is the information available to the market participants at the time $t = 0, \ldots, T$. Let us start deterministically $P(A) \in \{0, 1\}$ for all $A \in \mathcal{F}_0$ and restrict ourselves to consider adapted processes. Assume that $\mathcal{E} = \mathcal{E}^{phys} \cup \mathcal{E}^{fin}$ is a finite set of tradeable assets, where $\mathcal{E}^{phys}$ denotes physical assets (production capacity agreements) and $\mathcal{E}^{fin}$ stands for financial assets. Let $(R_t)_{t=1}^T$ be an $\mathbb{R}^{\mathcal{E}}$–valued process describing revenues, where $R_t(e)$ is the revenue from holding the asset $e \in \mathcal{E}$ within $[t-1, t]$. Suppose that $I \in \mathbb{N}$ agents may share the assets. An agent $i = 1, \ldots, I$ is determined by (x_i, U_i), where $x_i \in]0, \infty[$ denotes its initial endowment and U_i is its utility function

$$(2) \qquad U_i \in \{U \in C^1]0, \infty[: \ U' \text{ is positive, strictly decreasing with } \lim_{z \to \infty} U_i'(z) = 0\}.$$

At times $\{0, 1, 2, \ldots, T\}$ the agents $i = 1, \ldots, I$ trade the assets, which are arbitrarily divisible, moreover, short positions are allowed. At the end of each period, agents obtain their part of revenues and re–allocate their wealth. We agree to write $(\widehat{F}_t = F_t/N_t)_{t=0}^T$ for asset prices $(F_t)_{t=0}^T$ expressed in the units of savings security whose price process we denote by $(N_t)_{t=0}^T$. Under additional assumptions, we calculate equilibrium asset prices for given

$$(3) \qquad (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P), \ (N_t)_{t=0}^T \ (R_t)_{t=1}^T, (U_i, x_i)_{i=1}^I.$$

Let us explain the notion of equilibrium we use. Write $S_t = (S_t(e))_{e \in \mathcal{E}}$ to denote the price vector of all physical and financial assets $e \in \mathcal{E}$ at time t. A trading strategy $((\theta_t, \vartheta_t))_{t=0}^T$ determines the number θ_t of savings security units and the part $\vartheta_t(e)$ of each asset $e \in \mathcal{E}$ held by the agent within $]t, t+1]$. The strategy $((\theta_t, \vartheta_t))_{t=0}^T$ is called self–financed, if

$$(4) \qquad X_{t+1} = X_t + \theta_t(N_{t+1} - N_t) + \vartheta_t \circ (S_{t+1} - S_t + R_{t+1}) \qquad \text{for all } t = 0, \ldots, T-1,$$

where $(X_t = \theta_t N_t + \vartheta_t \circ S_t)_{t=0}^T$ denotes the wealth of this strategy. Let us point out that a self–financed $((\theta_t, \vartheta_t))_{t=0}^T$ is uniquely determined by its initial wealth $x = \theta_0 N_0 + \vartheta_0 \circ S_0$ and asset positions $\vartheta = (\vartheta_t)_{t=0}^T$. In fact, savings security positions $(\theta_t)_{t=0}^T$ are reconstructed from

(x, ϑ) recursively by

$$\theta_t = (X_t^{x,\vartheta,S} - \vartheta_t \circ S_t)/N_t, \tag{5}$$

$$X_{t+1}^{x,\vartheta,S} = X_t^{x,\vartheta,S} + \theta_t(N_{t+1} - N_t) + \vartheta_t \circ (S_{t+1} - S_t + R_{t+1}) \tag{6}$$

starting at $t = 0$ with $X_0^{x,\vartheta,S} = x$. Thus, each element from

$$\{(x, \vartheta) : \ x \in]0, \infty[, \ \vartheta = (\vartheta_t)_{t=0}^T \text{ is } (\mathcal{F}_t)_{t=0}^T\text{–adapted}\} \tag{7}$$

corresponds by (5) and (6) to a unique self–financed strategy implying that the set (7) gives a parameterization of all self–financed strategies and obviously $(X_t^{x,\vartheta,S})_{t=0}^T$ is the wealth of the strategy determined by (x, ϑ). For given initial endowment $x \in]0, \infty[$, asset prices $S = (S_t)_{t=0}^T$, and utility function U, introduce admissible positions by

$$\mathcal{A}(x, S, U) := \{\vartheta = (\vartheta_t)_{t=0}^{T-1} : \widehat{X}_t^{x,\vartheta,S} \geq 0, \quad t = 0, \dots, T, \quad E(U(\widehat{X}_T^{x,\vartheta,S})^-) < \infty\}.$$

We suppose that each agent behaves rationally: given prices S, the agent chooses strategy $\vartheta^* \in \mathcal{A}(x, S, U)$ which maximizes $\mathcal{A}(x, S, U) \to \mathbb{R}, \quad \vartheta \mapsto E(U(\widehat{X}_T^{x,S,\vartheta}))$.

Definition 1. *An equilibrium* $(S^*, \vartheta^{1*}, \dots, \vartheta^{I*})$ *of electricity market with agents* $(x_i, U_i)_{i=1}^I$ *consists of price process* S^* *and agent's positions* $(\vartheta^{i*})_{i=1}^I$ *such that market clears as*

$$\sum_{i=1}^I \vartheta_t^{i*}(e) = 1 \text{ for all } e \in \mathcal{E}^{phys}, \qquad \sum_{i=1}^I \vartheta_t^{i*}(e) = 0 \text{ for all } e \in \mathcal{E}^{fin}, \ t = 0, \dots, T,$$

and ϑ^{i*} *maximizes* $\vartheta \mapsto E(U_i(\widehat{X}_T^{x_i,\vartheta,S^*}))$ *on* $\mathcal{A}(x_i, S^*, U_i)$ *for each* $i = 1, \dots, I$.

To ensure the existence of the equilibrium, we upgrade our analysis by additional assumptions.

Assumption 1: *The one–period revenue is integrable and bounded from below:*

$$E(|\widehat{R}_t(e)|) < \infty, \quad \text{essinf } \widehat{R}_t(e) > -\infty \qquad \text{for all } e \in \mathcal{E}^{phys}, \ t = 1, \dots, T. \tag{8}$$

Assumption 2: *All contracts lose their values at the final date:*

$$S_T(e) = 0 \qquad \text{for all } e \in \mathcal{E}. \tag{9}$$

Assumption 3: *All agents* $(x_i, U_i)_{i=1}^I$ *are equal: There exists a utility function* U *and an initial endowment* $x \in]0, \infty[$ *such that*

$$U_i = U, \quad x_i = x \quad \text{for all } i = 1, \dots, I. \tag{10}$$

The first assumption is reasonable since the capacity holder runs the unit if electricity price covers variable costs of production, otherwise, the unit is idle and causes merely fixed costs. Hence, one-period loss is bounded by the fixed costs aggregated within one period. In the second assumption, $\operatorname{essinf} \widehat{R}_t(e) > -\infty$ is justified if we suppose that agents trade contracts which are valid for the period $[0, T]$ and so the market prices vanish as the agreements expire. The third assumption helps to avoid an exact description of agents endowments and their utility functions which are not observed in reality. As we assumed that all agents are equal, it is naturally to suppose that they hold the same positions ϑ^*:

$$\vartheta^*_t(e) = 1/I \text{ for all } e \in \mathcal{E}^{phys}, \qquad \vartheta^*_t(e) = 0, \text{ for all } e \in \mathcal{E}^{fin} \quad \text{for all } t = 0, \dots, T. \tag{11}$$

Such an equilibrium $(S^*, \vartheta^*, \dots, \vartheta^*)$ is called symmetric. In the last section we show that a symmetric equilibrium exists, provided all agents are sufficiently wealthy (in the sense of (81)). Moreover, there is a measure Q equivalent to P such that equilibrium asset prices are given by their future revenues, expected with respect to Q:

$$\widehat{S}^*_t(e) = E_Q\Big(\sum_{u=t+1}^{T} \widehat{R}_u(e) \,|\mathcal{F}_t\Big), \qquad t = 0, \dots, T, \quad e \in \mathcal{E}. \tag{12}$$

The following approach is based on equilibrium, utilizing, however, merely the existence of Q. Note, however, that we can not replace equilibrium consideration by no–arbitrage arguments, since postulating an arbitrage–free asset dynamics does not explain the mechanism of price formation. Moreover, choosing some arbitrage–free discrete–time price dynamics, we end up merely with the *existence of some martingalizing measure* and get into difficulty with non–uniqueness of arbitrage–free valuation. A convenient way to price all contracts by the same measure in some sense *chosen by the market* is mathematically covered by the above equilibrium concept.

3 Interest rate formulation

In the previous section, we have outlined that mild assumptions ensure the existence of equilibrium and provide contract valuation by (12). This formula suggests that equilibrium pricing is performed by the equivalent–measure–methodology common for financial modeling. If the market data (3) with (8) – (10) are given, then we are able to explicitly price electricity contracts. However, in reality, most quantities in (3) are not known, instead, one usually observes exchange prices for various financial products. Hence, to overcome the unknown quantities in (12), we have to describe the asset dynamics directly under Q such that the observed exchange prices are explained as best as possible. Let us see how to proceed in this way for the case of power forward market. We focus on forward contracts with a fixed pre–specified delivery duration $\Delta > 0$.

Suppose there is an electricity market with (3) satisfying (8)– (10) where all agents are sufficiently wealthy as stated in (81). Choose the domestic currency as follows:

(13) currency unit at t is 1 MWh, constantly delivered within the interval $[t, t+\Delta]$.

Suppose that the savings security $(N_t)_{t=0}^T$ is a bank account in EURO paying a constant interest rate $r > 0$, which means that

(14)

$e^{-rt}N_t$ is the reciprocal EURO-price at time t for electricity delivered within $[t, t+\Delta]$.

In the symmetric equilibrium, there exists a measure Q such that the market price $p_t(\tau)$ at time t for power forward maturing at τ is given by

$$p_t(\tau) = N_t E_Q(\frac{1}{N_\tau}|\mathcal{F}_t) \quad \tau = 0, \ldots, T, \quad t = 0, \ldots, \tau.$$

since due to the numeraire (13), all power forward prices finish at one: $p_t(t) = 1$ for all $t = 0, \ldots, T$, so we describe their dynamics using interest rate theory. To do so, we apply Heath–Jarrow–Morton (HJM) formulation for *spot martingale measure*, which assumes that the *wealth of the self financing strategy investing entirely in just maturing bonds* is the standard numeraire security and supposes that all asset prices, expressed in units of this numeraire follow martingales with respect to the spot martingale measure (see [4], [18]). Thus, to accomplish the analogy of electricity market to money market in the above form, we have to choose the *wealth of the self financing strategy investing entirely in just maturing power forwards* as the new numeraire. For this reason, we introduce the *sliding MWh* $(B_t)_{t=0}^T$ defined by

(15) $$B_t = \Pi_{u=1}^t p_{u-1}(u)^{-1}, \qquad t = 0, \ldots, T$$

which mimics the wealth of this strategy. Choosing $(B_t)_{t=0}^T$ as numeraire, we have to change from Q to the spot martingale measure $\tilde{Q}$ by

(16) $$d\tilde{Q} := \frac{N_0 B_T}{N_T B_0} dQ$$

in order to ensure the martingalizing property:

(17) for each process $(F_t)_{t=0}^T$ such that $(F_t/N_t)_{t=0}^T$ is a Q-martingale, $(F_t/B_t)_{t=0}^T$ is a $\tilde{Q}$-martingale.

As a consequence to this, the discounted electricity forward prices

$$(\widehat{p}_t(\tau) := p_t(\tau)/B_t)_{t=0}^{\tau} \qquad \text{are } \tilde{Q}\text{–martingales for all } \tau = 0, \dots, T. \tag{18}$$

Moreover, the discounted savings security

$$(\widehat{N}_t := N_t/B_t)_{t=0}^{T} \quad \text{is a } \tilde{Q}\text{–martingale.} \tag{19}$$

Now we turn to valuation of the European call option with strike price of K EURO and maturity date τ_1 written on the price of a power forward maturing at τ_2 where $0 \le \tau_1 \le \tau_2 \le T$ and $\tau_1, \tau_2 \in \{0, \dots, T\}$. The crucial point is that option strike price is given in EURO, which is a *foreign currency* in the electricity market. Thus, at time t, the European call price in EURO is

$$C_t = B_t E_{\tilde{Q}}((p_{\tau_1}(\tau_2) - Ke^{-r\tau_1}N_{\tau_1})^+ B_{\tau_1}^{-1} \,|\, \mathcal{F}_t)/(e^{-rt}N_t) \tag{20}$$

$$= B_t E_{\tilde{Q}}((\widehat{p}_{\tau_1}(\tau_2) - Ke^{-r\tau_1}\widehat{N}_{\tau_1})^+ \,|\, \mathcal{F}_t)/(e^{-rt}N_t) \tag{21}$$

Here $Ke^{-r\tau_1}N_{\tau_1}$ is the strike price in MWh, $B_{\tau_1}^{-1}, B_t$ occur due to discounting and undiscounting, and the division by $e^{-rt}N_t$ transforms from MWh back to EURO. We shall point out that the call option price is connected to the put option price

$$C_t^{\text{put}} = B_t E_{\tilde{Q}}((Ke^{-r\tau_1}N_{\tau_1} - p_{\tau_1}(\tau_2))^+ B_{\tau_1}^{-1} \,|\, \mathcal{F}_t)/(e^{-rt}N_t) \tag{22}$$

by the so–called put–call parity (see [18], p. 123)

$$C_t - C_t^{\text{put}} = p_t(\tau_2)/(e^{-rt}N_t) - e^{-r(\tau_1 - t)}K \tag{23}$$

which is obtained from the difference of (20) and (22) using the identity $(a-b)^+ - (b-a)^+ = a - b$. Note that the put price is uniquely determined by call price from (23), since $p_t(\tau_2)/(e^{-rt}N_t)$ is the *observed* EURO price of the underlying forward at the time t.

For the reminder of this section, we substantiate the formula (21) by purposing $\tilde{Q}$–dynamics for $(N_t)_t$, $(p_t(\tau))_t$ and $(B_t)_t$ within the Heath–Jarrow–Morton framework. Therefore, we need an adequate model restatement in continuous time.

Let $(W_t)_{t\in[0,T]}$ be a d–dimensional Brownian motion given on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \tilde{Q})$ where the filtration is right–continuous, $\tilde{Q}$–completed version of the Brownian filtration. Suppose (see assumptions HJM.1, HJM.2 from [18]) that the power forward prices in MWh are given by

$$p_t(\tau) = \exp(-\int_t^\tau f_t(u)du) \quad \text{for all } t = [0, \tau], \tau \in [0, T] \tag{24}$$

where the forward rates $(f_t(\tau))_{t\in[0,\tau]}$ for each $\tau \in [0,T]$ fulfill

$$f_t(\tau) = f_0(\tau) + \int_0^t \alpha_u(\tau)du + \int_0^t \sigma_u(\tau)dW_u \tag{25}$$

with some Borel measurable function $f_0(\cdot) : [0,T] \to \mathbb{R}$ and coefficients

$$\alpha : \mathcal{D} \times \Omega \to \mathbb{R}, \qquad \sigma : \mathcal{D} \times \Omega \to \mathbb{R}^d$$

defined on $\mathcal{D} = \{(u,t) \,:\, 0 \le u \le t \le T\}$ such that $(\alpha_u(t))_{u\in[0,t]}$, $(\sigma_u(t))_{u\in[0,t]}$ are for each $t \in [0,T]$ adapted processes satisfying

$$\int_0^t \alpha_u(t)du < \infty, \quad \int_0^t \sigma_u(t)^2 du < \infty \quad \text{for all } t \in [0,T].$$

In analogy to (15), we introduce

$$B_t := \exp(\int_0^t f_u(u)du) \quad t \in [0,T] \tag{26}$$

and according to (18), require that

$$(\widehat{p}_t(\tau) := p_t(\tau)/B_t)_{t\in[0,\tau]} \quad \text{are } \tilde{Q}\text{–maritingales for all } \tau \in [0,T]. \tag{27}$$

Let us emphasize that due to (27), α is uniquely determined by σ:

$$\alpha_t(\tau) = \sigma_t(\tau) \int_t^\tau \sigma_t(u)du \qquad \text{for all } (t,\tau) \in \mathcal{D}. \tag{28}$$

Indeed, using (24), (25), and the Proposition 2.2.1 from [4] we have

$$d\widehat{p}_t(\tau) = \widehat{p}_t(\tau)\big(b_t(\tau) + \frac{1}{2}\|s_t(\tau)\|^2\big)dt + \widehat{p}_t(\tau)s_t(\tau)dW_t \tag{29}$$

with coefficients

$$b_t(\tau) = -\int_t^\tau \alpha_t(u)du, \qquad s_t(\tau) = -\int_t^\tau \sigma_t(u)du \qquad \text{for all } t \in [0,\tau],\ t \in [0,T]. \tag{30}$$

Thus, (27) and (29) are combined to conclude that $b_t(\tau) + \frac{1}{2}\|s_t(\tau)\|^2 = 0$ for all $(t,\tau) \in \mathcal{D}$. Differentiating this equation with respect to τ we obtain (28), which with (29) finally provides a substantiated version of (18):

$$d\widehat{p}_t(\tau) = \widehat{p}_t(\tau)s_t(\tau)dW_t \qquad \text{for all } \tau \in [0,T]. \tag{31}$$

The continuous–time counterpart of positive $\tilde{Q}$–martingale (19) will be an exponential $\tilde{Q}$–martingale, due to the Brownian framework:

$$d\widehat{N}_t = \widehat{N}_t v_t dW_t. \tag{32}$$

Now we sketch the use of HJM modeling following [4]. Specify (this is a modeling part) the volatilities

$$(33) \qquad (v_t)_{t\in[0,T]}, \qquad (\sigma_t(\tau))_{t\in[0,\tau]} \quad \text{for } \tau \in [0,T]$$

and today's forward rate curve

$$f_0(\tau) = -\frac{\partial}{\partial\tau}\ln p_0(\tau) \qquad \tau \in [0,T].$$

Then $f_t(\tau)$ for $(t,\tau) \in \mathcal{D}$ is determined by (25) where $\int_0^t \alpha_u(\tau)du$ is fixed in (28), and power forward prices are found according to (24). We may now price electricity options. Let us illustrate how it works by valuing European call written on a power forward. To obtain a closed–form solution, we focus on deterministic volatility structures.

Proposition 1. *Suppose that*

$$(34) \qquad s_t(\tau) - v_t \text{ is deterministic for all } t \in [0,\tau] \text{ and } \tau \in [0,T],$$

then EURO–price C_t at the time $t \in [0,T]$ for European call with strike price $K > 0$ EURO, time to maturity $\tau_1 \in [t,T]$ written on power forward with time to maturity $\tau_2 \in [\tau_1,T]$ is given by

$$(35) \qquad C_t = P_t(\tau_2)\mathcal{N}(d(1)) - e^{-r(\tau_1-t)}K\mathcal{N}(d(2))$$

where $P_t(\tau_2)$ is the EURO–price at the time t for the underlying forward and

$$\begin{aligned} d(1) &= \frac{1}{\Sigma}(\ln(\frac{P_t(\tau_2)}{K}) + r(\tau_1-t) + \frac{1}{2}\Sigma^2) \\ d(2) &= d(1) - \Sigma, \\ (36) \qquad \Sigma^2 &= \Sigma^2(t,\tau_1,\tau_2) = \int_t^{\tau_1} \|s_u(\tau_2) - v_u\|^2 du. \end{aligned}$$

Proof. Using the new measure

$$dQ' = \frac{\widehat{N}_{\tau_1}}{\widehat{N}_t} d\tilde{Q},$$

we rewrite the formula (21) as

$$\begin{aligned} C_t &= B_t E_{\tilde{Q}}(\widehat{N}_{\tau_1}(\frac{\widehat{p}_{\tau_1}(\tau_2)}{\widehat{N}_{\tau_1}} - e^{-r\tau_1}K)^+ \,|\, \mathcal{F}_t)/(e^{-rt}N_t) \\ &= B_t E_{Q'}(\widehat{N}_t(\frac{\widehat{p}_{\tau_1}(\tau_2)}{\widehat{N}_{\tau_1}} - e^{-r\tau_1}K)^+ \,|\, \mathcal{F}_t)/(e^{-rt}N_t) \\ &= e^{rt} E_{Q'}((\frac{\widehat{p}_{\tau_1}(\tau_2)}{\widehat{N}_{\tau_1}} - e^{-r\tau_1}K)^+ \,|\, \mathcal{F}_t) \\ (37) \qquad &= E_{Q'}((e^{rt}\frac{\widehat{p}_{\tau_1}(\tau_2)}{\widehat{N}_{\tau_1}} - e^{-r(\tau_1-t)}K)^+ \,|\, \mathcal{F}_t). \end{aligned}$$

Introduce

$$P_u(\tau_2) = e^{ru}\frac{\widehat{p}_u(\tau_2)}{\widehat{N}_u} = e^{ru}\frac{p_u(\tau_2)}{N_u} = p_u(\tau_2)(e^{-ru}N_u)^{-1} \quad \text{for all } u \in [0,\tau_1]$$

which is in view of (14) interpreted as the EURO price at time u for electricity delivered within $[\tau_2, \tau_2 + \Delta]$. Define

$$\mathcal{E}_u(\tau_2) = e^{-ru}P_u(\tau_2) = \frac{p_u(\tau_2)}{N_u} \qquad u \in [0,\tau_1]$$

which possesses the stochastic differential

$$d\mathcal{E}_u(\tau_2) = \mathcal{E}_u(\tau_2)\big(\gamma_u du + \beta_u dW_u\big) \tag{38}$$

where Ito formula yields coefficients

$$\gamma_u = \|v_u\|^2 - v_u s_u(\tau_2), \quad \beta_u = s_t(\tau_2) - v_u \quad \text{for all } u \in [0,\tau_1]. \tag{39}$$

Let us write the solution to (38) as

$$\begin{aligned} \mathcal{E}_u(\tau_2) &= \mathcal{E}_0(\tau_2)e^{L_u - \frac{1}{2}[L]_u} && \text{for all } u \in [0,\tau_1] \\ &= \mathcal{E}_t(\tau_2)\exp\big(L_u - L_t - \frac{1}{2}([L]_u - [L]_t)\big) && \text{for all } u \in [t,\tau_1] \end{aligned} \tag{40}$$

where

$$L_u = \int_0^u \gamma_q dq + \int_0^u \beta_q dW_q, \qquad [L]_u = \int_0^u \|\beta_q\|^2 dq \qquad \text{for all } u \in [0,\tau_1]. \tag{41}$$

With these quantities, (37) reads as

$$C_t = E_{Q'}((P_t(\tau_2)\exp(L_{\tau_1} - L_t - \frac{1}{2}([L]_{\tau_1} - [L]_t) - e^{-r(\tau_1 - t)}K)^+ \,|\, \mathcal{F}_t). \tag{42}$$

According to Girsanov theorem, $(L_u)_{u\in[0,\tau_1]}$ follows a continuous martingale under Q'. Using the deterministic time change

$$l(u) = \inf\{q \in [0,\tau_1] \,:\, [L]_q > u\} \quad \text{for all } u \in [0,[L]_{\tau_1}]$$

we verify with time–change properties for continuous semimartingales (see [16], Theorem 4.6, chapter 3) that

$$W'_u := L_{l(u)}, \quad \mathcal{F}'_u := \mathcal{F}_{l(u)} \qquad \text{for all } u \in [0,\tau_1]$$

defines a Q'–Brownian motion $(W'_u, \mathcal{F}'_u)_{u\in[0,\tau_1]}$, satisfying

$$L_u = W'_{[L]_u} \quad \text{almost surely for all } u \in [0,\tau_1]. \tag{43}$$

Since the quadratic variation $[L]$ is deterministic,

$$G := L_{\tau_1} - L_t = W'_{[L]_{\tau_1}} - W'_{[L]_t} \tag{44}$$

follows due to (36), (39), and (41) under Q' a centered Gaussian distribution with variance

$$E_{Q'}(G^2) = [L]_{\tau_1} - [L]_t = \Sigma^2(t, \tau_1, \tau_2).$$

Thus, we obtain from (42) with (43) and (44) that

$$C_t = E_{Q'}(\big(P_t(\tau_2)\exp(G - \frac{1}{2}\Sigma^2) - e^{-r(\tau_1 - t)}K\big)^+ \, |\mathcal{F}_t). \tag{45}$$

Being an increment, G is Q'–independent from $\mathcal{F}'_{[L]_t} = \mathcal{F}_{l([L]_t)}$ and also Q'–independent from $\mathcal{F}_t \subseteq \mathcal{F}_{l([L]_t)}$ where the inclusion holds due to $t \leq l([L]_t)$. The expression (35) follows now from (45) by a straight–forward derivation. □

Let us point out that C_t in (42) is alternatively determined using Black–Scholes formula

$$C_t = \mathrm{BS}(P_t(\tau_2), K, \tau_1, t, r, \sqrt{\Sigma^2(t, \tau_1, \tau_2)/(\tau_1 - t)}) \tag{46}$$

with $\Sigma^2(t, \tau_1, \tau_2)$ from (36) and

$$\mathrm{BS}(s, k, \tau, t, r, \sigma) \quad := \quad s\mathcal{N}(d_+) - e^{-r(\tau - t)}k\mathcal{N}(d_-) \tag{47}$$

with

$$d_+ = \frac{1}{\sigma\sqrt{\tau - t}}(\ln(\frac{s}{k}) + (r + \frac{1}{2}\sigma^2)(\tau - t)), \qquad d_- = d_+ - \sigma\sqrt{\tau - t}.$$

That is, given model parameters, we obtain the *plug–in volatility* as

$$\varphi(t, \tau_1, \tau_2) = \sqrt{\Sigma^2(t, \tau_1, \tau_2)/(\tau_1 - t)}, \qquad (t, \tau_1), (\tau_1, \tau_2) \in \mathcal{D} \tag{48}$$

to price electricity options using Black–Scholes formula as in (46). In principle, the relation (47) could also be utilized for the *implied calibration* of the model: calculating implied Black–Scholes volatilities for observed electricity option prices, we obtain $\varphi(t, \tau_1, \tau_2)$ for different times t, τ_1, τ_2 to adjust the model parameters. However, an application of this technique to option data from the power exchange NordPool did not yield satisfactory results. We have considered European calls written on seasonal forward contracts (all of them are exercised on the third Thursday of the month before the first delivery month of the underlying forward, see [23]). Analyzing the implied volatilities, we observe, up to few exceptions, a steplike implied volatility graph with a significant drop near maturity. Still, this observation yields not more evidence than that option prices are eventually settled using Black–Scholes formula with decreasing volatility near maturity date.

On the contrary to implied calibration, the *historical calibration* is based on a considerable amount of reliable historical forward prices. The next section discusses this method.

4 Historical calibration

An essential part of our pricing approach to disentangle the forward prices in MWh from effects caused by MWh–EURO fluctuation. That is, to fit the model to historical forward prices, we focus in the first step on power forwards in MWh, hereafter we establish connection to EURO prices. Let us exemplarily illustrate this procedure for the case of one–factor model.

We begin with the first step. Suppose that we have chosen the simplest one–factor model of forward rate dynamics specifying in (25) the constant and deterministic forward rate volatility $\sigma \in]0, \infty[$. Taking into account (28), we have thus

$$\sigma_t(\tau) = \sigma, \quad \alpha_t(\tau) = \sigma^2(\tau - t), \quad \text{for all } (t, \tau) \in \mathcal{D},$$

giving the forward rates in (25), and the forward prices in (24) as

$$f_t(\tau) = f_0(\tau) + \sigma^2 t(\tau - \frac{t}{2}) + \sigma W_t^1, \qquad (t, \tau) \in \mathcal{D} \tag{49}$$

$$p_t(\tau) = \frac{p_0(\tau)}{p_0(t)} \exp(-\frac{\sigma^2}{2} t\tau(\tau - t) - \sigma(\tau - t)W_t^1), \qquad (t, \tau) \in \mathcal{D}. \tag{50}$$

Notice that due to (26) and (49) the dynamics of forward contracts in MWh is uniquely determined by the initial forward rate curve $f_0(\cdot)$ and by the forward rate volatility σ. Moreover, the sliding MWh is given by

$$B_t = \exp(\int_0^t f_u(u)du) = \exp(\int_0^t f_0(u)du + \frac{\sigma^2}{2}\frac{t^3}{3} + \sigma \int_0^t W_u^1 du), \qquad t \in [0, T]. \tag{51}$$

whereas the discounted forward prices in (31) are calculated with (50) and (51) as

$$\widehat{p}_t(\tau) = p_0(\tau) \exp(-\int_0^t \sigma(\tau - u)dW_u^1 - \frac{1}{2}\int_0^t |\sigma(\tau - u)|^2 du) \quad \text{for all } (t, \tau) \in \mathcal{D}. \tag{52}$$

Let us explain how to extract the model ingredients $f_0(\cdot)$, σ from market data, taking into account that in reality, the historical contract prices will be in EURO, which we denoted in what follows by

$$P_t(\tau) := \frac{p_t(\tau)}{e^{-rt}N_t} = \frac{\widehat{p}_t(\tau)}{e^{-rt}\widehat{N}_t} \qquad (t, \tau) \in \mathcal{D}. \tag{53}$$

The forward rate curve $f_0(\cdot)$ could, in principle, be red off from the initially observed EURO prices

$$f_0(t) = -\frac{\partial}{\partial t} \ln p_0(t) = -\frac{\partial}{\partial t} \ln P_0(t),$$

whereas the estimation of σ requires some additional considerations. Assume that we are given historical EURO–prices

$$P_{t_i}(T_1)(\omega), \quad P_{t_i}(T_2)(\omega), \quad \text{for } i = 0, \ldots, n. \tag{54}$$

at discrete times $t_0, \dots, t_n \in [0, T_1]$ with $0 \le t_0 < t_2 < \dots < t_n$ for power forwards maturing at $T_1 < T_2$ respectively. Adapting ideas from [20] to our context, we may estimate σ by the following procedure:

Lemma 1. *For the one–factor model as above and historical data (54), the maximum likelihood estimator of σ^2 is given by*

$$\overline{\sigma}^2 = \frac{-n + \sqrt{n^2 + 4\big(\sum_{i=0}^{n-1} b_i^2\big)\big(\sum_{i=0}^{n-1} a_i^2\big)}}{2\sum_{i=0}^{n-1} b_i^2} \tag{55}$$

where for $i = 0, \dots, n-1$ we put

$$b_i = \frac{(T_1 + T_2)(t_{i+1} - t_i) - (t_{i+1}^2 - t_i^2)}{2\sqrt{t_{i+1} - t_i}}, \tag{56}$$

$$a_i = \frac{\ln\big(P_{t_{i+1}}(T_1)/P_{t_{i+1}}(T_2)\big) - \ln\big(P_{t_i}(T_1)/P_{t_i}(T_2)\big)}{(T_2 - T_1)\sqrt{t_{i+1} - t_i}}(\omega). \tag{57}$$

Proof. To avoid forward prices in MWh, we consider the fraction

$$p_t(T_1)/p_t(T_2) = P_t(T_1)/P_t(T_2) \quad \text{for all } t \in [0, T_1], \tag{58}$$

whose values at $t \in \{t_0, \dots, t_n\}$ are available since the numerator and the denominator on the right side of (58) are known from (54). The explicit forward price formula (50) is combined with (58) to obtain

$$\begin{aligned}\frac{P_t(T_1)}{P_t(T_2)} &= \frac{P_0(T_1)}{P_0(T_2)} \exp\Big(-\frac{\sigma^2}{2} t(T_1(T_1 - t) - T_2(T_2 - t)) - \sigma W_t^1((T_1 - t) - (T_2 - t))\Big) \\ &= \frac{P_0(T_1)}{P_0(T_2)} \exp\Big(\frac{\sigma^2}{2} t(T_2 - T_1)(T_1 + T_2 - t) + (T_2 - T_1)\sigma W_t^1\Big).\end{aligned}$$

Thus, (57) is calculated as a realization sequence $(a_i = A_i(\omega))_{i=0}^{n-1}$ of independent random variables

$$\begin{aligned}A_i &:= \frac{\ln\big(P_{t_{i+1}}(T_1)/P_{t_{i+1}}(T_2)\big) - \ln\big(P_{t_i}(T_1)/P_{t_i}(T_2)\big)}{(T_2 - T_1)\sqrt{t_{i+1} - t_i}} \\ &= \sigma^2 \frac{(T_1 + T_2)(t_{i+1} - t_i) - (t_{i+1}^2 - t_i^2)}{2\sqrt{t_{i+1} - t_i}} + \sigma \frac{W_{t_{i+1}}^1 - W_{t_i}^1}{\sqrt{t_{i+1} - t_i}}, \quad i = 0, \dots, n-1.\end{aligned} \tag{59}$$

Here each A_i is $N(\sigma^2 b_i, \sigma^2)$–distributed with unknown parameter σ^2, hence, the logarithmic likelihood function of $A_0, \dots, A_{n-1}$ on the realization $(a_0, \dots, a_{n-1})$ is

$$L :]0, \infty[\to \mathbb{R}, \quad \sigma^2 \mapsto -n \ln(\sqrt{2\pi}) - n\frac{1}{2} \ln \sigma^2 - \sum_{i=0}^{n-1} \frac{(a_i - \sigma^2 b_i)^2}{2\sigma^2},$$

and its unique maximum is attained at (55). □

Now we turn to the second step estimating the behavior of $(N_t)_{t\in[0,T]}$. Since $(B_t)_{t\in[0,T]}$ is given by (51), the process $(N_t = B_t\widehat{N}_t)_{t\in[0,T]}$ is specified by the initial value $\widehat{N}_0 = N_0 = P_0(0)^{-1}$, (being the reciprocal EURO electricity price at time $t=0$ for electricity delivered within $[0,\Delta]$), and the volatility $(v_t)_{t\in[0,T]}$ in (32). Again, let us consider the simplest case where a positive parameter $v\in]0,\infty[$ is chosen to represent the constant and deterministic volatility. Moreover, we introduce a correlation parameter $\rho\in[-1,1]$ to capture the dependence between $\widehat{p}_t(\tau)$ and $\widehat{N}_t$ choosing for (32) the dynamics

$$d\widehat{N}_t = \widehat{N}_t v(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2), \qquad \widehat{N}_0 := P_0(0)^{-1}, \tag{60}$$

where $(W_t^2)_{t\in[0,T]}$ is a Brownian motion independent from $(W_t^1)_{t\in[0,T]}$. The solution to the above equation is written as

$$\widehat{N}_t = \widehat{N}_0 \exp(vV_t - \frac{v^2}{2}t), \quad t\in[0,T] \tag{61}$$

with another Brownian motion $(V_t := \rho W_t^1 + \sqrt{1-\rho^2}W_t^2)_{t\in[0,T]}$. In this settings, we are able to estimate the volatility v from historical data. However, to obtain an explicit formula, we restrict ourselves to equidistant–time observations

$$P_{t_j}(T_1)(\omega), \quad P_{t_j}(T_2)(\omega) \qquad j\in J := \{i\in\{0,\dots,n-1\} : t_{i+1}-t_i=\delta\} \tag{62}$$

chosen from (54) for a fixed time step $\delta>0$, (δ is, for example, one day).

Lemma 2. *Assume we are given σ in the one–factor model (50), and suppose the dynamics (60) with undetermined v and ρ. Define*

$$\overline{g}^2 := 2\frac{-|J| + \sqrt{|J|^2 + |J|\delta\sum_{j\in J}(c_j + \sigma^2\delta^{\frac{5}{2}}/24)^2}}{|J|\delta} \tag{63}$$

where for $i=0,\dots,n-1$ we put

$$\begin{aligned} c_i &:= (t_{i+1}-t_i)^{-\frac{1}{2}}\big(\ln\frac{P_{t_{i+1}}(T_1)}{P_{t_i}(T_1)}(\omega) - r(t_{i+1}-t_i) \\ &\quad +\sigma^2\frac{(T_1-t_i)^3-(T_1-t_{i+1})^3}{6} + d_i\frac{(T_1-t_i)^2-(T_1-t_{i+1})^2}{2(t_{i+1}-t_i)}\big) \end{aligned} \tag{64}$$

$$\begin{aligned} d_i &:= \big(\ln\frac{P_{t_{i+1}}(T_1)}{P_{t_{i+1}}(T_2)}(\omega) - \ln\frac{P_{t_i}(T_1)}{P_{t_i}(T_2)}(\omega)\big)(T_2-T_1)^{-1} \\ &\quad -\frac{\sigma^2}{2}((T_1+T_2)(t_{i+1}-t_i) - (t_{i+1}^2 - t_i^2)) \end{aligned} \tag{65}$$

If $\overline{g}^2 > \frac{\sigma^2\delta^2}{12}$, then the maximum–likelihood estimator of v^2 based on observations (62) is given by

$$\overline{v}^2 = \overline{g}^2 - \frac{\sigma^2\delta^2}{12} \tag{66}$$

Proof. Suppose have shown that $(c_j)_{j\in J}$ is a realization sequence

$$(67)\qquad \begin{array}{l}(c_j = C_j(\omega))_{j\in J}, \text{ with } (C_j)_{j\in J} \text{ independent} \\ \text{identically } N(\frac{\sqrt{\delta}}{2}v^2, v^2 + \frac{\sigma^2\delta^2}{12})\text{–distributed,}\end{array}$$

then the logarithmic likelihood function of $(C_j)_{j\in J}$ on the realization $(c_j)_{j\in J}$ is $L(v^2 + \frac{\sigma^2\delta^2}{12})$ where we use the variable transform $g^2 = v^2 + \frac{\sigma^2\delta^2}{12}$ to define

$$L :]0,\infty[\to \mathbb{R}, \quad g^2 \mapsto -|J|\ln(\sqrt{2\pi}) - |J|\frac{1}{2}\ln(g^2) - \sum_{j\in J}\frac{(c_j + \sigma^2\delta^{5/2}/24 - g^2\sqrt{\delta}/2)^2}{2g^2},$$

attaining its unique maximum at $\overline{g}^2$ from (63). If $\overline{g}^2 - \frac{\sigma^2\delta^2}{12} > 0$, then this value will also be the unique maximum of $v^2 \mapsto L(v^2 + \frac{\sigma^2\delta^2}{12})$ on $]0,\infty[$ showing the assertion.

For later use, we need that a deterministic square integrable function h satisfies

$$(68)\qquad E(\int_{t_i}^{t_{i+1}} h(u)dW_u^1 \,|\, W_{t_0}^1,\ldots,W_{t_n}^1) = \overline{h}_i(W_{t_{i+1}}^1 - W_{t_i}^1) \qquad i = 0,\ldots,n-1$$

where $\overline{h}_i = \int_{t_i}^{t_{i+1}} h(u)du/(t_{i+1}-t_i)$ is the average of the function h on $[t_i, t_{i+1}]$. Moreover, the differences

$$R_i := \int_{t_i}^{t_{i+1}} h(u)dW_u^1 - E(\int_{t_i}^{t_{i+1}} h(u)dW_u^1 \,|\, W_{t_0}^1,\ldots,W_{t_n}^1) \qquad i = 0,\ldots,n-1$$

are centered Gaussian with

$$(69)\qquad E(R_i^2) = \int_{t_i}^{t_{i+1}} |h(u) - \overline{h}_i|^2 du \text{ and } (R_i, W_{t_{i+1}}^1 - W_{t_i}^1)_{i=0}^{n-1} \text{ are independent.}$$

We complete the proof by verifying (67). From (53), we conclude that

$$(70)\qquad \ln\frac{P_{t_{i+1}}(T_1)}{P_{t_i}(T_1)} = \ln\frac{\widehat{p}_{t_{i+1}}(T_1)}{\widehat{p}_{t_i}(T_1)} - \ln\frac{\widehat{N}_{t_{i+1}}}{\widehat{N}_{t_i}} + r(t_{i+1} - t_i)$$

where in view of (52), the first summand on the right gives

$$(71)\qquad \ln\frac{\widehat{p}_{t_{i+1}}(T_1)}{\widehat{p}_{t_i}(T_1)} = -\int_{t_i}^{t_{i+1}} \sigma(T_1 - u)dW_u^1 - \frac{1}{2}\int_{t_i}^{t_{i+1}} |\sigma(T_1 - u)|^2 du$$

Form (68), calculate the conditional expectation

$$(72)\, E(\int_{t_i}^{t_{i+1}} \sigma(T_1 - u)dW_u^1 \mid W_{t_0}^1,\ldots,W_{t_n}^1) = \sigma(W_{t_{i+1}}^1 - W_{t_i}^1)\frac{(T_1 - t_i)^2 - (T_1 - t_{i+1})^2}{2(t_{i+1} - t_i)}$$

to rewrite (71) as

$$(73)\qquad \begin{aligned}\ln\frac{\widehat{p}_{t_{i+1}}(T_1)}{\widehat{p}_{t_i}(T_1)} &= -\sigma(W_{t_{i+1}}^1 - W_{t_i}^1)\frac{(T_1 - t_i)^2 - (T_1 - t_{i+1})^2}{2(t_{i+1} - t_i)} - R_i \\ &\quad -\frac{\sigma^2}{2}\frac{(T_1 - t_i)^3 - (T_1 - t_{i+1})^3}{3}\end{aligned}$$

where R_i equals to $\int_{t_i}^{t_{i+1}} \sigma(T_1 - u) dW_u^1$ less its conditional expectation (72), ensuring with (69) that

$$\begin{aligned} &\{R_i, W^1_{t_{i+1}} - W^1_{t_i}, W^2_{t_{i+1}} - W^2_{t_{i+1}} \, : \, i = 0, \dots, n-1\} \text{ are independent,} \\ &R_i \text{ is } N(0, \tfrac{\sigma^2}{12}(t_{i+1} - t_i)^3)\text{–distributed for all } i = 0, \dots, n-1. \end{aligned} \tag{74}$$

The increment $\sigma(W^1_{t_{i+1}} - W^1_{t_i})$ is obtained by the arguments of (59) as

$$\sigma(W^1_{t_{i+1}} - W^1_{t_i}) = \big(\ln \frac{P_{t_{i+1}}(T_1)}{P_{t_{i+1}}(T_2)} - \ln \frac{P_{t_i}(T_1)}{P_{t_i}(T_2)}\big)(T_2 - T_1)^{-1} - \frac{\sigma^2}{2}\big((T_1 + T_2)(t_{i+1} - t_i) - (t^2_{i+1} - t^2_i)\big)$$

showing that

$$d_i = D_i(\omega) \qquad \text{where } D_i := \sigma(W^1_{t_{i+1}} - W^1_{t_i}) \text{ for all } i = 0, \dots, n-1. \tag{75}$$

Combine now (70), (73), and the previous equation to deduce that $(c_i = C_i(\omega))_{i=0}^{n-1}$ where

$$C_i = \frac{-\ln\big(\widehat{N}_{t_{i+1}}/\widehat{N}_{t_i}\big) - R_i}{\sqrt{t_{i+1} - t_i}} = \frac{v^2}{2}\sqrt{t_{i+1} - t_i} - v\frac{V_{t_{i+1}} - V_{t_i}}{\sqrt{t_{i+1} - t_i}} - \frac{R_i}{\sqrt{t_{i+1} - t_i}}. \tag{76}$$

Due to (74), $C_0, \dots, C_{n-1}$ are independent and each C_i follows Gaussian distribution with mean $v^2\sqrt{t_{i+1} - t_i}/2$ and variance $v^2 + \sigma^2(t_{i+1} - t_i)^2/12$ which shows (67) according to $\delta = (t_{i+1} - t_i)$ for all $i \in J$. □

Given model parameters $\sigma \in]0, \infty[$ for (50) and $v \in]0, \infty[$ for (61), we estimate the unknown correlation $\rho = dW_t^1 dV_t$, from observations (62) using

$$\bar{\rho} := \frac{|J|^{-1}\sum_{j \in J} e_j d_j}{\sqrt{|J|^{-1}\sum_{j \in J}(e_j^2 - \frac{\sigma^2\delta^3}{12})}\sqrt{|J|^{-1}\sum_{j \in J} d_j^2}} \tag{77}$$

where $(d_i)_{i=0}^{n-1}$ are from (65) and for $i = 0, \dots, n-1$

$$\begin{aligned} e_i = \frac{v^2}{2}(t_{i+1} - t_i) - \big(&\ln \frac{P_{t_{i+1}}(T_1)}{P_{t_i}(T_1)}(\omega) - r(t_{i+1} - t_i) \\ &+ \sigma^2 \frac{(T_1 - t_i)^3 - (T_1 - t_{i+1})^3}{6} + d_i \frac{(T_1 - t_i)^2 - (T_1 - t_{i+1})^2}{2(t_{i+1} - t_i)}\big). \end{aligned}$$

Indeed, (64) and (76) yield

$$e_i = E_i(\omega) \qquad \text{where } E_i = v(V_{t_{i+1}} - V_{t_i}) + R_i \;\; i = 0, \dots, n-1 \tag{78}$$

showing with (75) and (74) that each series $(E_j)_{j \in J}$, $(D_j)_{j \in J}$ and $(E_j D_j)_{j \in J}$ is a sequence of independent, identically distributed random variables with

$$E(E_j^2) = v^2\delta + \sigma^2\delta^3/12, \quad E(D_j^2) = \sigma^2\delta, \quad E(D_j E_j) = \sigma v \rho \delta \quad \text{for all } j \in J.$$

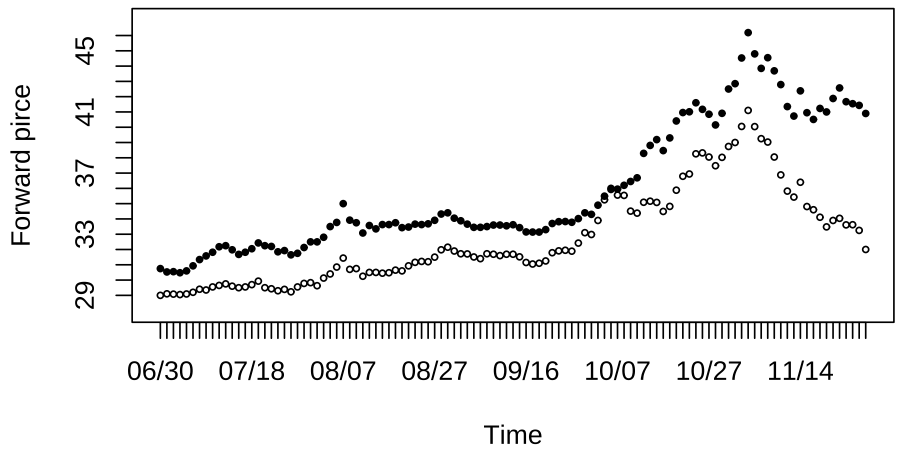

Figure 1: The prices for December 2003 and January 2004, denoted by $\circ$ and $\bullet$ respectively

Thus, for large sample size $|J| \to \infty$ the estimate (77) converges almost surely towards ρ by the strong law of large numbers.

Example Let us estimate the parameters $\sigma,\ v,\ \rho$ from historical market prices. The data used in this example are publicly available at `http://www.eex.de`, from the European Energy Exchange (EEX) in Leipzig, where next to power for physical delivery, financial forward and future contracts are traded. To find reasonable estimators, it is important to have a sufficiently long data series. On this account we will consider forward contracts with a monthly delivery period, which are traded six months before maturity and thus have a reasonable time series. Within one month, a standardized base load contract supplies an amount of electrical energy determined by

$$(24 \times \text{ Number of days within the delivery month) MWh}$$

with the exception of March (743 MWh) and October (745 MWh). However, for commensurability reasons, the prices are listed daily in EURO for 1 MWh of energy delivered as a constant flow over the corresponding month, being the so–called final settlement price, see [24], p. 14. Our data are specified as follows: $\Delta =$ one month, $T_1 =$ 1st of December 2003, $T_2 =$ 1st of January 2004. The time set where both forward contracts are traded consists of 109 daily price pairs as in (54) which are illustrated in Figure 1. The observations start at 30th of June 2003 and finish at 28th of December 2003. The parameter σ^2 is estimated using the equation (55) and all observed forward prices $P_{t_i}(T_1)$, $P_{t_i}(T_2)$, $i = 0, \dots, n$. To estimate other parameters, we took the subset $J \subset \{t_0, \dots, t_n\}$ where for each $i \in J$ the time to the next observation is $t_{i+1} - t_i = \delta := 1/365 =$ one day. This leaves us with $|J| = 87$ observed price pairs giving an estimation of v^2 and ρ according to the formulas (66) and (77) respectively. The results are summarized in the Table 1.

Parameter	ML Estimate
σ	2.063
v	0.5177
ρ	-0.7754

Table 1: Parameter estimates

Now we turn to option valuation. Since the volatilities are deterministic, EURO–prices for European calls are calculated from (35) with Σ^2 as in (36)

$$\begin{aligned}\Sigma^2(t, \tau_1, \tau_2) &= \int_t^{\tau_1} \left(|-\sigma(\tau_2 - u) - \rho v|^2 + |v\sqrt{1-\rho^2}|^2\right) du \\ &= \sigma^2 \frac{(\tau_2 - t)^3 - (\tau_2 - \tau_1)^3}{3} + 2\sigma\rho v \frac{(\tau_2 - t)^2 - (\tau_2 - \tau_1)^2}{2} + v^2(\tau_1 - t). \qquad (79)\end{aligned}$$

Given estimated values of σ, v and ρ from the Table 1 we calculate the plug–in volatility

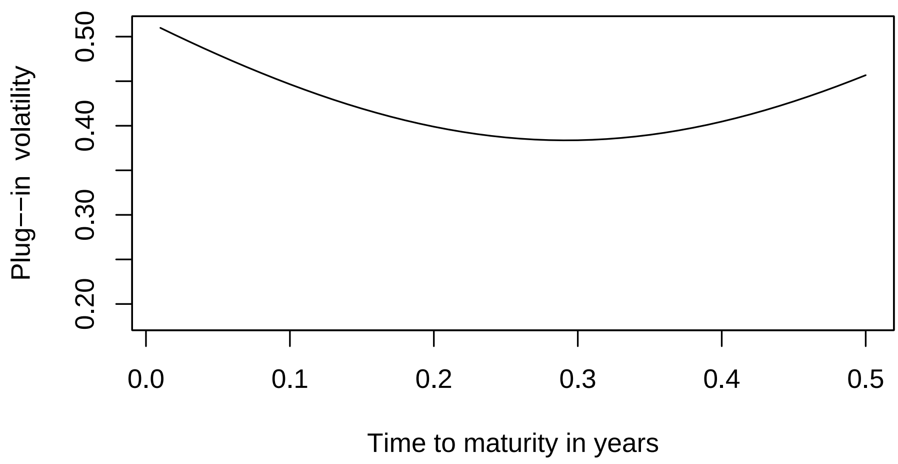

Figure 2: Plug–in volatility $\theta \mapsto \varphi(\tau_1 - \theta, \tau_1, \tau_2)$ for parameters from Table 1 and $\tau_1 = \tau_2 = 0.5$

$\varphi(t, \tau_1, \tau_2)$ from (79) and (48). Figure 2 illustrates the dependence $\theta \mapsto \varphi(\tau_1 - \theta, \tau_1, \tau_2)$ of this volatility on the time to maturity $\theta \in [0, \tau_1]$ for the most interesting case where option's expiry date $\tau_1 = 0.5$ coincides with underlying's delivery start $\tau_2 = 0.5$.

5 Appendix

Let us show that if the agent's endowment is sufficiently large, then a symmetric equilibrium exist. We use the following notations:

$$\widehat{R} := \frac{1}{I} \sum_{e \in \mathcal{E}^{phys}} \sum_{t=1}^{T} \widehat{R}_t(e), \qquad \widehat{r} := \operatorname{essinf} \widehat{R} > -\infty$$

Proposition 2. *Consider an electricity market where the revenues $(R_t)_{t=1}^T$ fulfill (8) and agents $(x_i, U_i)_{i=1}^I$ satisfy (10) with endowment $x \in]0, \infty[$ and utility function U.*
(i) The mapping

$$s(\cdot) :]0, \infty[\to \mathbb{R}, \quad \varepsilon \mapsto \frac{E(U'(\varepsilon + \widehat{R} - \widehat{r})\widehat{R})}{E(U'(\varepsilon + \widehat{R} - \widehat{r}))} \tag{80}$$

defines a continuous function $s(\cdot)$ with $s(\varepsilon) \geq \widehat{r}$ for all $\varepsilon > 0$.
(ii) If the initial endowment $x \in]0, \infty[$ is sufficiently large in the sense that

$$x > \inf_{\varepsilon > 0} (\varepsilon + s(\varepsilon)) - \widehat{r}, \tag{81}$$

then there exists a solution $s^* \in \mathbb{R}$ to the equation

$$E\Big(\frac{U'(x - s + \widehat{R})}{E(U'(x - s + \widehat{R}))} \widehat{R}\Big) = s, \qquad s \in] - \infty, x + \widehat{r}[. \tag{82}$$

(iii) If (81) is fulfilled and s^* solves (82), then define a new probability measure Q by

$$dQ := \frac{U'(x - s^* + \widehat{R})}{E(U'(x - s^* + \widehat{R}))} dP. \tag{83}$$

The price process S^*, given by

$$\widehat{S}_t^*(e) = E_Q\Big(\sum_{j=t+1}^{T} \widehat{R}_j(e) \,|\mathcal{F}_t\Big), \qquad t = 0, \dots, T, \quad e \in \mathcal{E} \tag{84}$$

satisfies (9) and $(S^*, \vartheta^*, \dots, \vartheta^*)$ with ϑ^* from (11) is a symmetric equilibrium.

Proof. (i) For all $\varepsilon > 0$, the random variable $U'(\varepsilon + \widehat{R} - \widehat{r})\widehat{R}$ is integrable since

$$|U'(\varepsilon + \widehat{R} - \widehat{r})\widehat{R}| \quad \leq \quad U'(\varepsilon)|\widehat{R}|.$$

Moreover, for each sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset]0,\infty[$ converging to $\varepsilon > 0$, we have

$$\lim_{j\to\infty} U'(\varepsilon_j + \widehat{R} - \widehat{r})\widehat{R} = U'(\varepsilon + \widehat{R} - \widehat{r})\widehat{R}$$

almost sure dominated by the integrable $U'(\inf_{j\in\mathbb{N}} \varepsilon_j + \widehat{R} - \widehat{r})|\widehat{R}|$, which shows the continuity of the numerator $\varepsilon \mapsto E(U'(\varepsilon + \widehat{R} - \widehat{r})\widehat{R})$ in (80). The same arguments apply to show the continuity of the denominator $\varepsilon \mapsto E(U'(\varepsilon + \widehat{R} - \widehat{r}))$ which is positive since U' is strictly positive. This gives the continuity of $s(\cdot)$. To show that $s(\cdot)$ is bounded from below by $\widehat{r}$, we consider $U'(\varepsilon + \widehat{R} - \widehat{r})/E(U'(\varepsilon + \widehat{R} - \widehat{r}))$ as density of a new measure and interprete $s(\varepsilon)$ as the expectation of $\widehat{R}$ with respect to this new measure. Then $s(\varepsilon) \geq \widehat{r}$ is an immediate consequence of $\widehat{R} \geq \widehat{r}$.

(ii) If x satisfies (81), then there exist ε_0 with

$$x + \widehat{r} - \varepsilon_0 - s(\varepsilon_0) > 0. \tag{85}$$

Choosing $\varepsilon_1 > 0$ with $\varepsilon_1 > \varepsilon_0$ and $\varepsilon_1 > x$, we see that

$$x + \widehat{r} - \varepsilon_1 - s(\varepsilon_1) < 0, \tag{86}$$

since $\widehat{r} \leq s(\varepsilon_1)$. Now, the intermediate value theorem yields $\varepsilon^* \in]\varepsilon_0, \varepsilon_1[$ such that

$$x - s(\varepsilon^*) = \varepsilon^* - \widehat{r}. \tag{87}$$

By definition of the function $s(\cdot)$, we have $E(U'(\varepsilon^* + \widehat{R} - \widehat{r})(\widehat{R} - s(\varepsilon^*)) = 0$ and (87) gives

$$E(U'(x - s(\varepsilon^*) + \widehat{R})(\widehat{R} - s(\varepsilon^*))) = 0. \tag{88}$$

Set $s^* = s(\varepsilon^*)$ to rewrite the identity (88) equivalently as

$$E(\frac{U'(x - s^* + \widehat{R})}{E(U'(x - s^* + \widehat{R}))}\widehat{R}) = s^*. \tag{89}$$

(iii) From (84) which gives (9) where S is replaced by S^* we conclude that

$$\widehat{X}_T^{x,\vartheta^*,S^*} = x - \frac{1}{I}\sum_{e\in\mathcal{E}^{phys}} \widehat{S}_0(e) + \frac{1}{I}\sum_{e\in\mathcal{E}^{phys}}\sum_{t=1}^{T} \widehat{R}_t = x - s^* + \widehat{R} \tag{90}$$

where the last equality is obtained from (84) and (89) by verifying

$$\frac{1}{I}\sum_{e\in\mathcal{E}} \widehat{S}_0(e) = E(\frac{U'(x - s^* + \widehat{R})}{E(U'(x - s^* + \widehat{R}))}\widehat{R}|\mathcal{F}_0) = s^*.$$

We have the strict positivity and with (89) the integrability:

$$\widehat{X}_T^{x,\vartheta^*,S^*} = \varepsilon^* + \widehat{R} - \widehat{r} \geq \varepsilon^* > 0, \qquad E_Q(\widehat{X}_T^{x,\vartheta^*,S^*}) = x < \infty. \tag{91}$$

According to the definition (84),

$$(L_{t+1} := \widehat{S}^*_{t+1} - \widehat{S}^*_t + \widehat{R}_{t+1})_{t=0}^{T-1} \tag{92}$$

are Q–martingale increments, which gives the Q–martingale property of $(\widehat{X}_t^{x,S^*,\vartheta^*})_{t=0}^T$ due to the recursion

$$\widehat{X}_{t+1}^{x,\vartheta,S} \quad = \quad \widehat{X}_t^{x,\vartheta,S^1} + \vartheta_t(\widehat{S}_{t+1} - \widehat{S}_t + \widehat{R}_{t+1}), \quad \widehat{X}_0^{x,\vartheta,S} = x \qquad \text{for all } \vartheta \in \mathcal{A}(x,S,U).$$

Combining this with (91) we deduce

$$\widehat{X}_t^{x,\vartheta^*,S^*} = E_Q(\widehat{X}_T^{x,\vartheta^*,S^*}|\mathcal{F}_t) > 0 \qquad \text{for all } t = 0,\dots,T.$$

and so the admissibility $\vartheta^* \in \mathcal{A}(x,S^*,U)$ of the constant strategy ϑ^* follows with

$$E(U(\widehat{X}_T^{x,\vartheta^*,S^*})^-) < \infty$$

since $U(\widehat{X}_T^{x,\vartheta^*,S^*}) \geq U(\varepsilon^*)$ is bounded from below.

Let us show that for all $\vartheta \in \mathcal{A}(x,S^*,U)$ the wealth $(\widehat{X}_t^{x,\vartheta,S^*})_{t=0}^T$ is a Q–martingale. Given $\vartheta \in \mathcal{A}(x,S^*,U)$, define a bounded ϑ^M by

$$\vartheta_t^M(\omega) := \begin{cases} \vartheta_t(\omega) & \text{if } |\vartheta_t(\omega)| \leq M \\ M \cdot \operatorname{sign} \vartheta_t(\omega) & \text{if } |\vartheta_t(\omega)| > M \end{cases} \qquad \text{for all } t = 0,\dots,T-1$$

Then for all $t = 0,\dots,T-1$ we have the monotone convergence for both the positive and the negative part as

$$\lim_{M\to\infty} (\vartheta_t^M L_{t+1})^+ = (\vartheta_t L_{t+1})^+, \qquad \lim_{M\to\infty} (\vartheta_t^M L_{t+1})^- = (\vartheta_t L_{t+1})^-. \tag{93}$$

Moreover, since ϑ_t^M is bounded and L_{t+1} is a Q–martingale increment, we obtain

$$0 = E_Q(\vartheta_t^M L_{t+1}) = E_Q((\vartheta_t^M L_{t+1})^+) - E_Q((\vartheta_t^M L_{t+1})^-) \qquad \text{for all } M > 0. \tag{94}$$

Further, $\widehat{X}_t^{x,\vartheta,S^*} + \vartheta_t L_{t+1} = \widehat{X}_{t+1}^{x,S^*,\vartheta} \geq 0$ implies that $(\vartheta_t L_{t+1})^- \leq \widehat{X}_t^{x,\vartheta,S^*}$, hence

$$(\vartheta_t^M L_{t+1})^- \leq (\vartheta_t L_{t+1})^- \leq \widehat{X}_t^{x,\vartheta,S^*} \qquad \text{for all } M > 0. \tag{95}$$

If $X_t^{x,\vartheta,S^*} \in L^1(\Omega,\mathcal{F},Q)$, then it follows from (94) and (95) that

$$E_Q((\vartheta_t^M L_{t+1})^+) = E_Q((\vartheta_t^M L_{t+1})^-) \leq E_Q(\widehat{X}_t^{x,\vartheta,S^*}) \qquad \text{for all } M > 0.$$

and the monotone convergence in (93) ensures that $(\vartheta_t L_{t+1})^+, (\vartheta_t L_{t+1})^- \in L^1(\Omega,\mathcal{F},Q)$ and that the random variables $((\vartheta_t^M L_{t+1})^+)_{M>0}$ and $((\vartheta_t^M L_{t+1})^-)_{M>0}$ tend for $M \to \infty$ to $(\vartheta_t L_{t+1})^+$ and $(\vartheta_t L_{t+1})^-$ respectively in $L^1(\Omega,\mathcal{F},Q)$–sense which shows that

$$0 = \lim_{M\to\infty} E_Q(\vartheta_t^M L_{t+1}|\mathcal{F}_t) = E_Q(\vartheta_t L_{t+1}|\mathcal{F}_t).$$

Thus, for $\vartheta \in \mathcal{A}(x, S^*, U)$ we conclude by induction for all $t = 0, \ldots, T-1$ that $\widehat{X}^{x,\vartheta,S^*}_{t+1}$ is integrable with respect to Q and satisfies $E_Q(\widehat{X}^{x,\vartheta,S^*}_{t+1} | \mathcal{F}_t) = \widehat{X}^{x,\vartheta,S^*}_t$ giving the Q–martingale property for $(\widehat{X}^{x,\vartheta,S^*}_t)^T_{t=0}$.

Note that for each U from (2), the inverse function $J := U'^{\,-1}$ maps $]0, \sup_{z>0} U'(z)[$ onto $]0, \infty[$ and satisfies the inequality

$$U(J(b)) \geq U(a) + b(J(b) - a) \qquad \text{for all } a \in]0, \infty[, \quad b \in]0, \sup_{z>0} U'(z)[. \tag{96}$$

To see that the asset prices S^* belong to a symmetric equilibrium, we apply (96) with $a := \widehat{X}^{x,\vartheta,S^*}_T$ for arbitrary $\vartheta \in \mathcal{A}(x, S^*, U)$ and $b := U'(x - s^* + \widehat{R})$. By (90), we are led to

$$U(\widehat{X}^{x,\vartheta^*,S^*}_T) = U(J(U'(x - s^* + \widehat{R}))) \geq U(\widehat{X}^{x,\vartheta,S^*}_T) + U'(x - s^* + \widehat{R})(\widehat{X}^{x,\vartheta^*,S^*}_T - \widehat{X}^{x,\vartheta,S^*}_T).$$

We now calculate the expectation on both sides of the previous inequality taking (83) into account:

$$E(U(\widehat{X}^{x,\vartheta^*,S^*}_T)) \geq E(U(\widehat{X}^{x,\vartheta,S^*}_T)) + E_Q(\widehat{X}^{x,\vartheta^*,S^*}_T - \widehat{X}^{x,\vartheta,S^*}_T) E(U'(x - s^* + \widehat{R})).$$

The Q–martingale property yields $E_Q(\widehat{X}^{x,\vartheta^*,S^*}_T - \widehat{X}^{x,\vartheta,S^*}_T) = 0$, which implies

$$E(U(\widehat{X}^{x,\vartheta^* S^*}_T)) \geq E(U(\widehat{X}^{x,\vartheta,S^*}_T)) \qquad \text{for all } \vartheta \in \mathcal{A}(x, S^*, U) \tag{97}$$

showing that that for asset prices S^* the best strategy is in fact to hold the constant part $1/I$ of each physical asset and no financial asset positions. □

6 Conclusions

In this paper we purpose a new approach to pricing electricity derivatives. Our considerations show that taking into account the electricity production process, one justifies the applicability of efficient martingale methods to pricing arbitrary electricity contracts. Utilizing this concept for the case of electricity forward market, we are able to benefit from the machinery of interest rate theory by appropriate change–of–numeraire transformation. An application of this technique yields explicit formulas for European options written on forward contracts. Moreover, we show how to calibrate the models on historical prices discussing a one–factor model with constant forward rate. Even this simple framework exhibits a term structure of the implied Black–Scholes volatility, described by the notion of plug–in volatility in our settings. We conclude that a market participant purchasing European call option written on a power forward can not rely to sell that contract later on the same Black–Scholes volatility level. Similarly, an agent procuring electricity options, should keep in mind the impact of time–dependence from the plug–in volatility on option's Delta in order to achieve precise hedging strategies. We hope that the presented contribution will provide a quantitative framework to treat these and related problems.

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